Optical Signal Processing

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Chapter 4
Transmittance Functions, Lenses and Gratings

Introduction

- **The beam sources in previous lectures are simple apertures illuminated by a plane wave**.
- They are modeled with real functions and have a zero phase component.
- In this chapter, functions are presented that create a more complicated field by altering the magnitude and/or phase of the field.
- **In general, these transmittance functions can be thought of as multiplying an incident field to create a desired effect.**
- Some represent well-known optical components such as a lens or a diffraction grating.

Tilt

- **An optical beam can be steered in a propagation simulation by applying a "tilt" to the beam wavefront.**
- Suppose a tilt of angle of α is applied to a wavefront, as shown in Fig. 1.

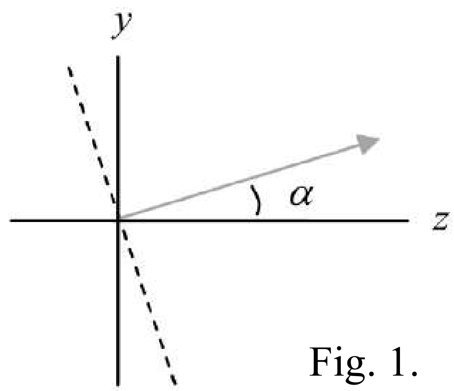
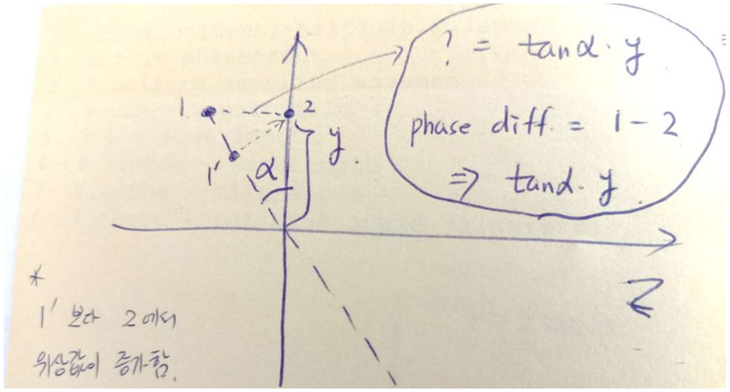
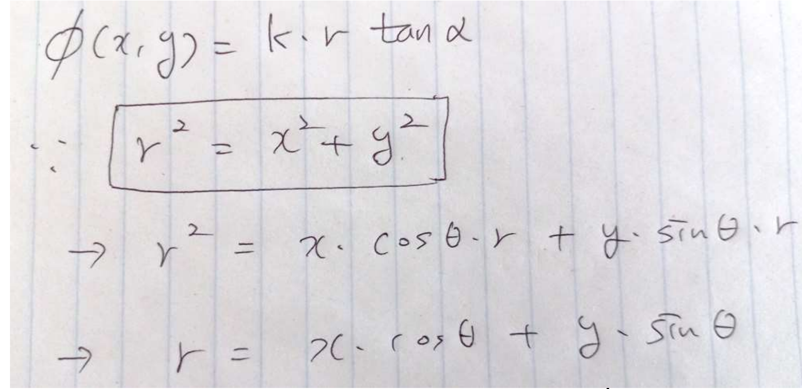


Fig. 1.

- where the dashed line represents the tilted wavefront of the beam and the arrow indicates the intended direction of propagation.

Tilt





- The wavelength λ corresponds to 2π rad in the phase notation, so using the wavenumber parameter $k = 2\pi/\lambda$, the phase function for producing the tilt is

$$\phi_Y(x, y) = ky \tan\alpha .$$

- **More generally, to produce a tilt α relative to the z axis but in a radial direction defined by the angle θ in the x–y plane (see Fig. 2), we can use**

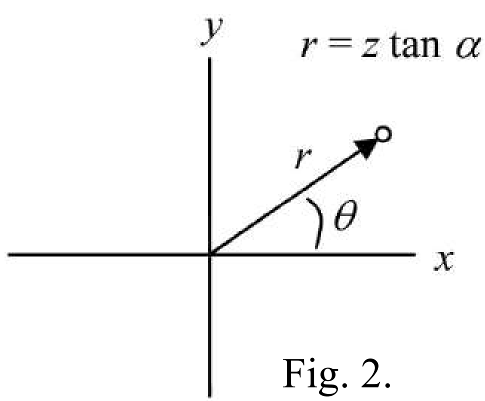


Fig. 2.

$$\phi(x, y) = k(x\cos\theta + y\sin\theta)\tan\alpha , \qquad \theta = \tan^{-1}(y/x).$$

Tilt

- The transmittance function is, therefore,

$$t_A(x, y) = \exp\left[jk(x\cos\theta + y\sin\theta)\tan\alpha\right].$$

- The function tilt is listed here:

```
function[uout]=tilt(uin,L,lambda,alpha,theta)
% tilt phasefront
% uniform sampling assumed
% uin - input field
% L - side length
% lambda - wavelength
% alpha - tilt angle
% theta - rotation angle (x axis 0)
% uout - output field

[M,N]=size(uin);          %get input field array size
dx=L/M;                   %sample interval
k=2*pi/lambda;            %wavenumber

x=-L/2:dx:L/2-dx;         %coords
[X,Y]=meshgrid(x,x);

uout=uin.*exp(j*k*(X*cos(theta)+Y*sin(theta))...
   *tan(alpha));         %apply tilt
end
```

Tilt

- Test the tilt function by returning to the square beam example from the previous lecture.
- Insert the following before the propagation call:

```
deg=pi/180;
alpha=5.0e-5; %rad
theta=45*deg;
[u1]=tilt(u1,L1,lambda,alpha,theta);
```

- Fig. 3 is the irradiance result after executing the example.

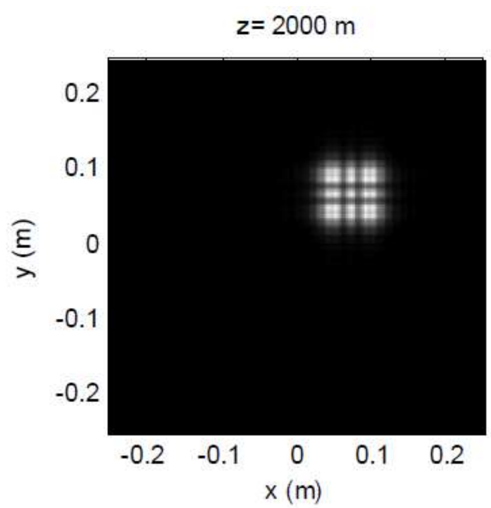


Fig. 3.

Figure 6.3 Irradiance pattern for the tilt example: α = 0.5 µrad, θ = 45 deg.

Tilt

- **<u>Sampling limitations also exist for this technique.</u>**
- As we might guess, if the tilt is large enough to translate the beam in the observation plane beyond the grid boundary, there will be trouble.
- To study this limitation, consider that tilt is a linear phase exponential applied to the source function U_1.
- **<u>Assume a single-axis tilt, and apply the shift theorem to the transform of the source field:</u>**

$$\Im\{U_1(x_1, y_1)\exp(jkx_1 \tan\alpha)\} = G_1\left(f_{X1} - \frac{\tan\alpha}{\lambda}, f_{Y1}\right),$$

- where $G_1 = \Im\{U_1\}$.

Tilt

- If the source has a bandwidth of B_1, then considering the spectrum is essentially shifted, the combined effective bandwidth for the field leaving the source is

$$B_1^{+T} = B_1 + \frac{\tan\alpha}{\lambda}. \tag{1}$$

- The propagation criterion is: $B_1^{+T} \leq 1/(2\Delta x)$.
- Using Eq. (1), and with some rearrangement, we get

$$\alpha \leq \lambda\left(\frac{1}{2\Delta x_1} - B_1\right),$$

- **<u>In typical simulations, the maximum available tilt angle is quite small, which is consistent with the paraxial nature of the Fresnel propagator.</u>**

Focus

- Another useful operation is converging ("focusing") or diverging ("defocusing") an optical beam.
- A beam with a spherical wavefront, as shown in Fig. 4, will converge to the position z_f on the z axis.

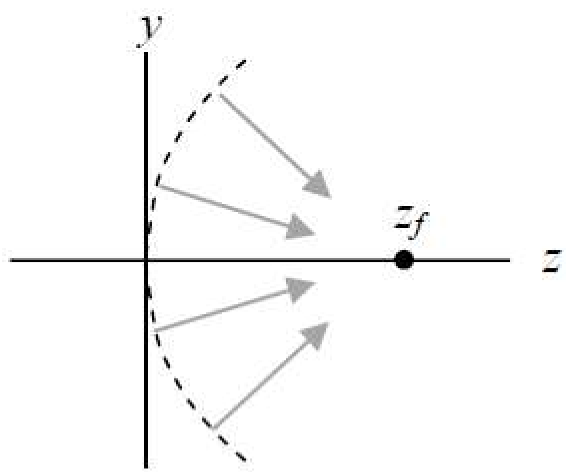
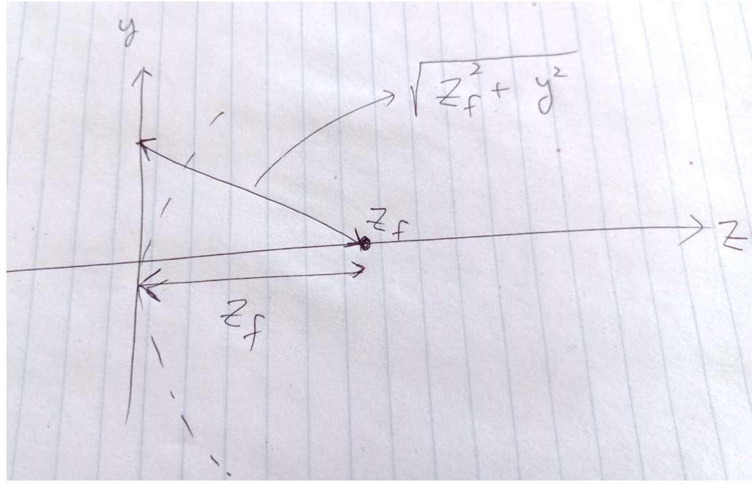


Fig. 4. Converging wavefront.

Focus



- We can proceed in the same manner as was done for tilt to find the converging phase front in the *x*–*y* plane at *z* = 0.
- This is given by $\phi_S(x,y) = -k\sqrt{z_f{}^2 + x^2 + y^2}$. (2)
- The negative sign in the above Eq. (2) corresponds to **a converging wavefront**, as illustrated in Fig. 4.
- A positive sign is for **a diverging wavefront**.

Focus

- The application of the binomial approximation gives a parabolic phase front that approximates the spherical phase front:

$$\phi(x, y) = -\frac{k}{2z_f}\left(x^2 + y^2\right).$$

- The transmittance function for focus is, therefore,

$$t_A(x, y) = \exp\left[-j\frac{k}{2z_f}\left(x^2 + y^2\right)\right]. \qquad (3)$$

- **This is a phase chirp function of the same form for the Fresnel impulse response function *h*, although the exponent sign is negative.**

Focus

- A MATLAB function for applying focus follows:

```
function[uout]=focus(uin,L,lambda,zf)
% converging or diverging phase-front
% uniform sampling assumed
% uin - input field
% L - side length
% lambda - wavelength
% zf - focal distance (+ converge, - diverge)
% uout - output field

[M,N]=size(uin);           %get input field array size
dx=L/M;                    %sample interval
k=2*pi/lambda;             %wavenumber
%
x=-L/2:dx:L/2-dx;          %coords
[X,Y]=meshgrid(x,x);

uout=uin.*exp(-j*k/(2*zf)*(X.^2+Y.^2)); %apply focus
end
```

Focus

- Try this on the square beam example.
- Insert the following before the propagation call and run the script:

```
zf=2000;
[u1]=focus(u1,L1,lambda,zf);
```

- The result is shown in Fig. 5.

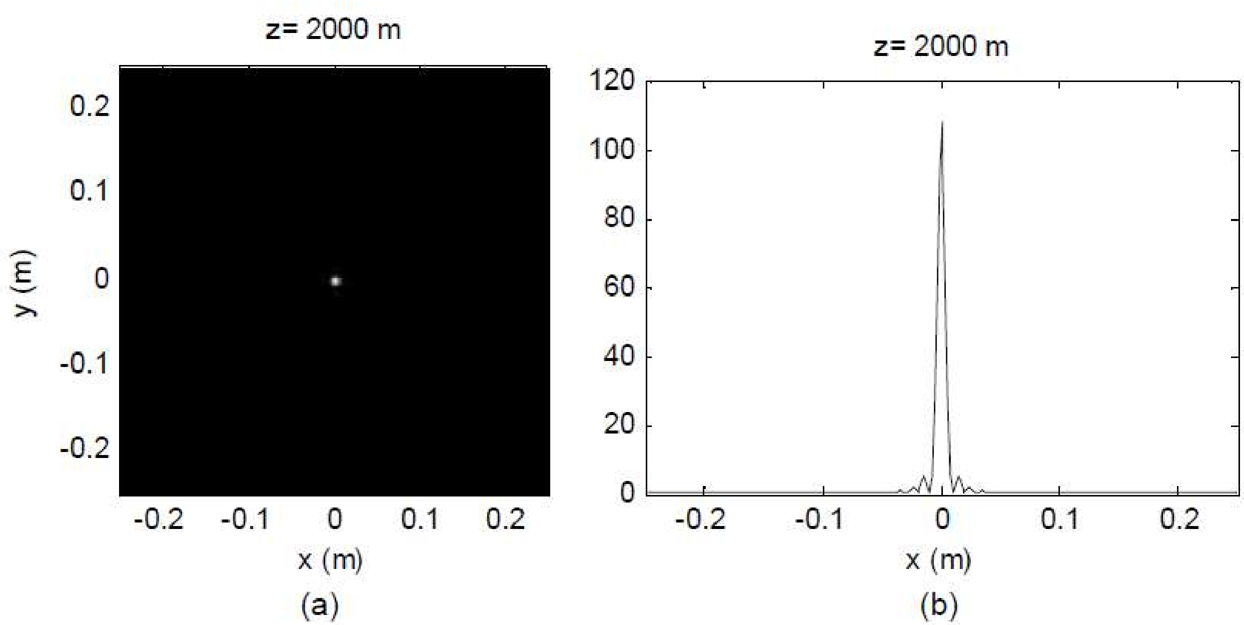
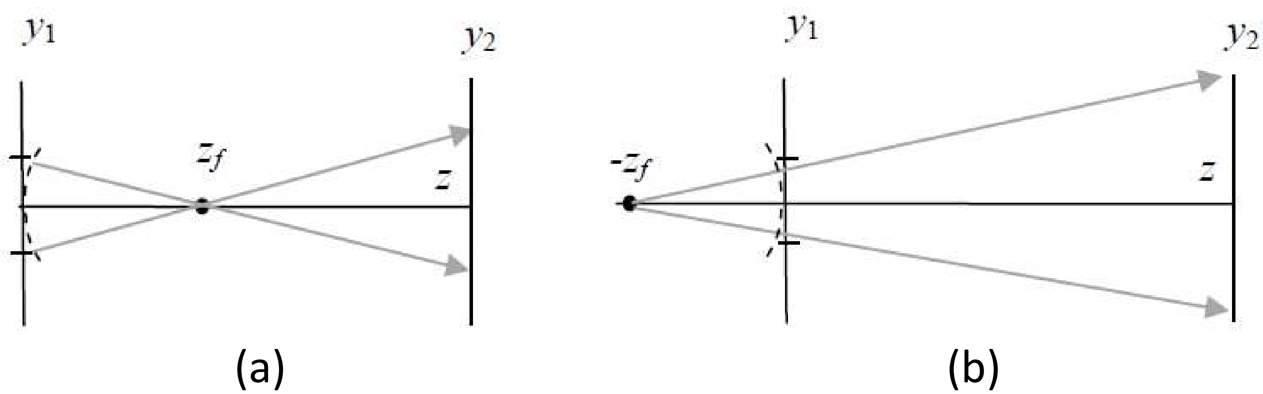


Fig. 5. (a) Image irradiance and (b) profile for the focus example.

Focus

- A negative focus value puts the focal point in a virtual position behind the plane and causes a diverging wave (see the following Fig. (b)).



Geometrical ray diagram for (a) a converging wavefront and (b) a diverging wavefront at the source plane.

Focus

- Multiplying a source field by Eq. (3) has the effect of increasing the source bandwidth.
- The combination effective bandwidth is approximately:

$$B_1^{+F} \approx B_1 + \frac{D_1/2}{\lambda |z_f|}.$$

- where D_1 is the effective support of the source field (maximum linear width)
- For $B_1^{+F} \leq 1/(2\Delta x)$, then with some rearranging, a bound is obtained for the focal distance:

$$|z_f| \geq \frac{D_1/2}{\lambda}\left(\frac{1}{2\Delta x_1} - B_1\right)^{-1}. \quad (4)$$

The cutoff frequency in the square (or circular) pupil function is:

$$f_0 = \frac{w_{XP}}{\lambda z_{XP}}.$$

where $w_{XP} = D/2$

Lens

- **A lens is an optical element that uses refraction to focus or diverge light.**
- **The transmittance function for an ideal, simple lens is given by:**

$$t_A(x,y) = P(x,y)\exp\left[-j\frac{k}{2f}\left(x^2+y^2\right)\right], \tag{5}$$

- where f is known as the focal length and $P(x,y)$ is the pupil function.

Lens

- **This is essentially the same complex exponential defined for focus with z_f replaced by f.**
- A positive focal length produces a converging wavefront from a plane-wave input and a negative focal length produces a diverging wavefront.
- The pupil function accounts for the physical size of the lens.
- **For example, the most common lens pupil function is a circle**

$$P(x,y) = \mathrm{circ}\left(\frac{\sqrt{x^2+y^2}}{w_L}\right),$$

- where w_L is the radius of the lens aperture.

Lens

- It is not always practical to implement the transmittance function of Eq. (5) in Fresnel propagation as was done for the focus example.
- **This is because the focal length *f* is governed by the same criterion as z_f given in Eq. (4), and since *f* tends to be relatively short, a large number of samples are required.**

Lens

- **Assume a plane wave incident on the lens, which implies $B_1 \rightarrow 0$, then with some algebra the expression in Eq. (4) leads to**

$$\frac{|f|}{D_L} \geq \frac{\Delta x}{\lambda},$$

- where $D_L = 2w_L$. The ratio $|f|/D_L$ is known as the focal ratio, or the *f-number*, indicated by *f /#.*
- With this substitution,

$$f/\# \geq \frac{\Delta x}{\lambda}. \tag{6}$$

Lens

- Practical lenses have *f /#* ranging from perhaps 2 to roughly 50 and diameters $2w_L$ of a few millimeters to maybe 100 mm.
- Take a typical value, *f/#* = 10 and a diameter of 25 mm.
- Assume visible light $\lambda = 0.5\times10^{-6}$ m, Eq. (6) yields

$$\Delta x \leq 5\times10^{-6}\ \mathrm{m}\,.$$

- Furthermore, to implement Fresnel propagation, the array side length L needs to at least span the lens diameter.

Lens

- Thus, the linear number of samples in an array required to model this lens is

$$M \geq \frac{L}{\Delta x} = \frac{25 \times 10^{-3}}{5 \times 10^{-6}} = 5000,$$

- Thus, modeling lenses directly with the Fresnel propagator is usually practical only for large *f/#*.
- However, all is not lost for smaller *f/#*.
- If the field incident on the lens is $U_1(x_1, y_1)$, then the field exiting the lens is $U_1(x_1, y_1)t_A(x_1, y_1)$.

Lens

- Insert this into the Fresnel diffraction expression and set *z = f.*
- The chirp functions in the integral cancel, and the result is

$$U_2(x_2, y_2) = \frac{\exp(jkf)}{j\lambda f} \exp\left[j\frac{k}{2f}\left(x_2^{\ 2} + y_2^{\ 2}\right)\right] \\ \times \iint U_1(x_1, y_1) P(x_1, y_1) \exp\left[-j\frac{2\pi}{\lambda f}\left(x_2 x_1 + y_2 y_1\right)\right] dx_1 dy_1. \tag{7}$$

- **The expression in Eq. (7) shows the field at the *focal plane* of an ideal positive lens is simply the Fraunhofer pattern of the incident field with *z = f.***

Lens

- **<u>To find the field or irradiance pattern in the focal plane of a positive lens, including one with a small *f/#,* the function "prop_FF" from the previous lecture can be applied replacing *z* with *f*.</u>**
- Take the parameters from the *f/*10 lens example and assume U_1 is a unit amplitude plane wave (select M = 250 and L = 250 mm*).*

Lens

- **<u>Figure 6 shows the focused irradiance pattern formed with an ideal circular-shaped lens, which is known as the Airy pattern.</u>**

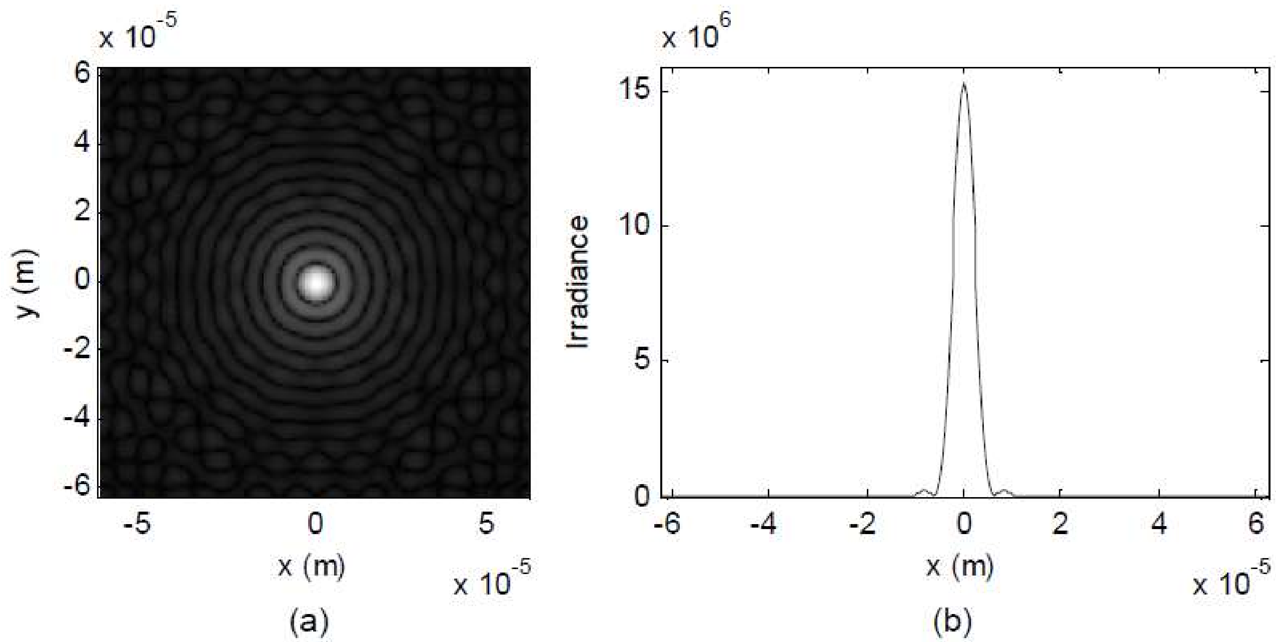


Fig. 6. (a) Image irradiance and (b) x-axis profile *for the lens Fraunhofer pattern.*
The large peak irradiance value in (b) is because all of the power in the unit amplitude field incident on the lens is being focused to a very small area.

Lens

- **<u>A special case of interest is when the source field is located in the front focal plane of a positive lens, a distance *f* from the lens (see Fig. 7).</u>**

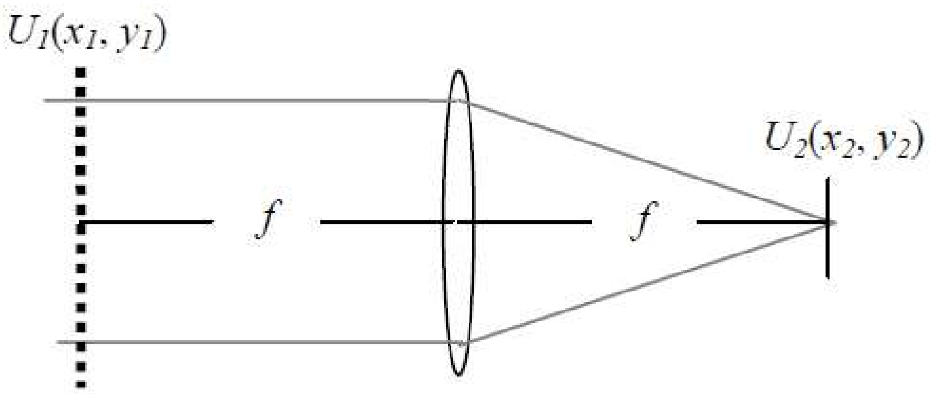


Fig. 7. "Fourier transform" lens arrangement.

- For this arrangement the field at the focal plane is (refer to Fourier Optics, figure 5.6)

$$U_2(x_2, y_2) = \frac{\exp(jkf)}{j\lambda f}$$

$$\times \iint U_1(x_1, y_1) P(x_1 + x_2, y_1 + y_2) \exp\left[-j\frac{2\pi}{\lambda f}(x_2 x_1 + y_2 y_1)\right] dx_1 dy_1.$$

Lens

- **The chirp phase factor out front is now gone, so the focal plane field is a scaled *Fourier transform* of the input field.**
- The arguments in the pupil function account for ***vignetting***, which is a loss of light for off-axis points in the input field due to the finite pupil size.
- **The effect of vignetting can be reduced if the lens pupil is oversized compared to the support of the input field.**

Gratings

- **<u>A grating is an optical component that has a spatially periodic structure.</u>**
- Incident light diffracts either in transmission or reflection from the structure, and the colors (wavelength components) of the light become spatially separated some distance from the grating.
- Gratings are commonly used in spectrometers for examining the wavelength spectrum of an optical signal or in spectrophotometers that measure the spectral characteristics of an optical component.
- The diffraction pattern from a grating is usually observed in the Fraunhofer region.

Gratings (Cosine Magnitude Example)

- A conventional grating has grooves cut into its surface that impart a magnitude and/or phase disturbance to the incident wave.
- **<u>To demonstrate modeling of periodic functions like those in gratings, start with an amplitude transmittance function given by:</u>**

$$t_A(x,y) = \frac{1}{2}\left[1 - \cos\left(2\pi \frac{x}{P}\right)\right] \mathrm{rect}\left(\frac{x}{D_1}\right) \mathrm{rect}\left(\frac{y}{D_1}\right). \qquad (8)$$

Gratings (Cosine Magnitude Example)

- Figure 8 illustrates a 1D profile of this grating.

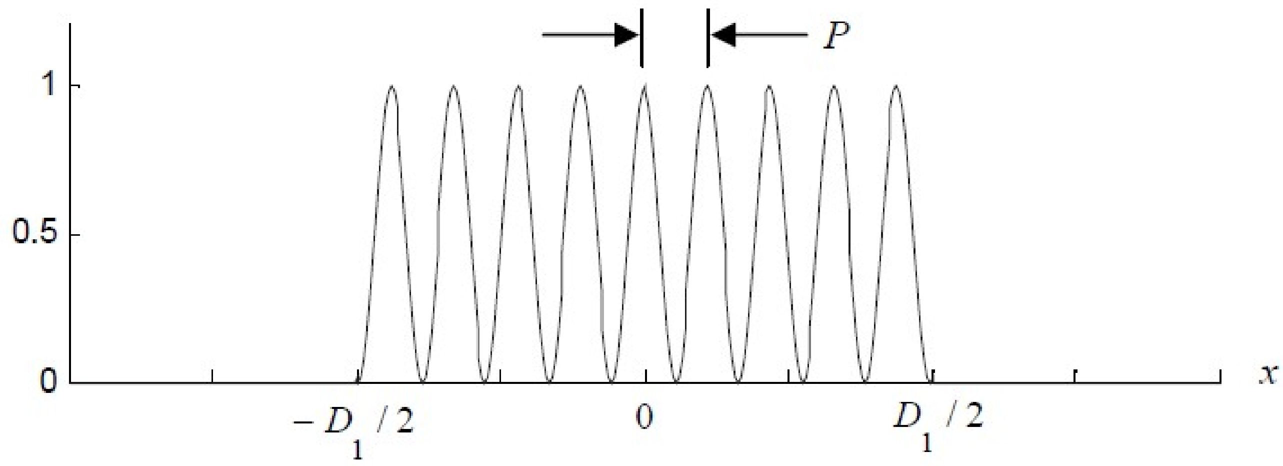


Fig. 8. Cosine grating profile.

Gratings (Cosine Magnitude Example)

- In Eq. (8) the grating is defined within the 2D area $D_1 \times D_1$.
- The cosine pattern is only a function of *x* and has a period *P.*
- Typically, $P << D_1$.
- When illuminated by a unit amplitude plane wave, the source field is $U_1(x_1, y_1)=t_A(x_1, y_1)$.
- The Fraunhofer pattern is created using a lens of focal length *f.*

Gratings (Cosine Magnitude Example)

- The grating is simulated in the following script:

```
% grating_cos diffraction grating example

lambda=0.5e-6;  %wavelength
f=0.5;          %propagation distance
P=1e-4;         %grating period
D1=1.02e-3;     %grating side length

L1=1e-2;        %array side length
M=500;          %# samples
dx1=L1/M;
x1=-L1/2:dx1:L1/2-dx1; %source coords
[X1,Y1]=meshgrid(x1,x1);

% Grating field and irradiance
u1=1/2*(1-cos(2*pi*X1/P)).*rect(X1/D1).*rect(Y1/D1);

% Fraunhofer pattern
[u2,L2]=propFF(u1,L1,lambda,f);
dx2=L2/M;
x2=-L2/2:dx2:L2/2-dx2; y2=x2;%obs coords
I2=abs(u2).^2;
```

Gratings (Cosine Magnitude Example)

- The periodic function should be sampled adequately.
- **<u>The number of samples that span the periodic function (the length *P*) is:</u>**

$$\#\,\text{samples across}\, P = M\,\frac{P}{L_1}.$$

- In case of the cosine grating, $MP/L_1 = 5$ indicates that five samples span each cosine cycle (it is ok).
- At least two are required to satisfy the sampling theorem.
- The lens focal length is chosen as f=0.5m.

Gratings (Cosine Magnitude Example)

- Figure 9 shows irradiance images of the source plane (I1) and the observation plane (I2) and an x-axis profile in the observation plane.
- **<u>The central feature is known as the zero order and the two side features are the −1 and +1 “first-order” peaks.</u>**

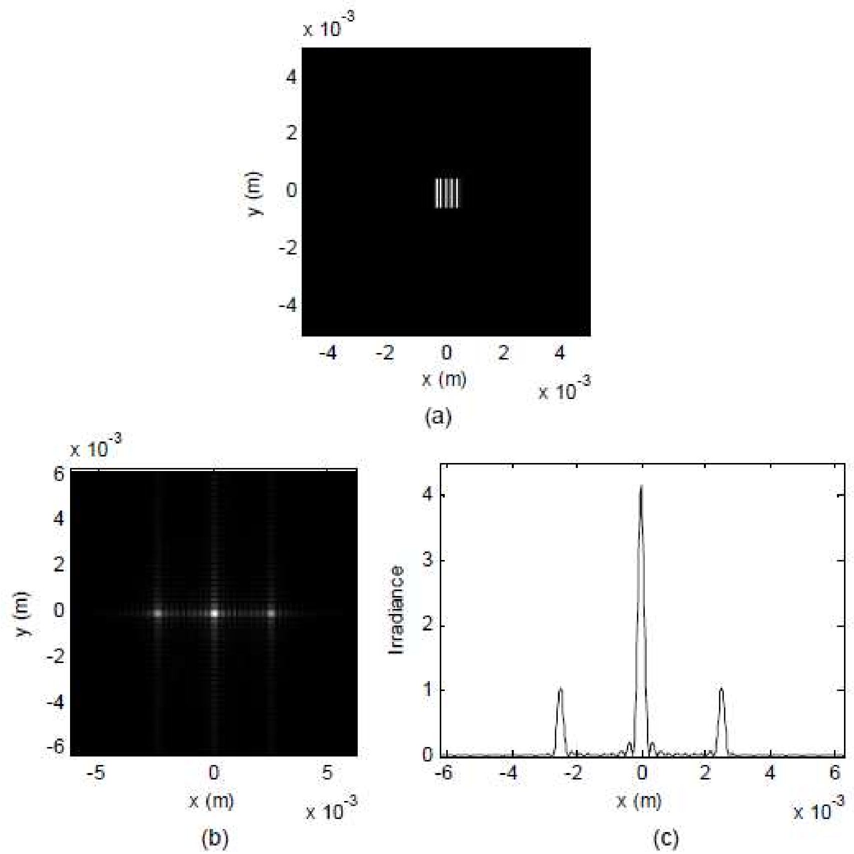


Fig. 9. Results for grating_cos: (a) source irradiance; (b) Fraunhofer irradance and (c) *x-axis profile of Fraunhofer irradiance.*

Gratings (Cosine Magnitude Example)

- To make sure the simulation is working properly, the results can be compared with the analytic expression for the Fraunhofer pattern.
- First, the Fourier transform of the source field is required:

$$\Im\{U_1(x_1,y_1)\} = \frac{1}{2}\left[\delta(f_{X1},f_{Y1}) - \frac{1}{2}\delta\left(f_{X1}-\frac{1}{P},f_{Y1}\right) - \frac{1}{2}\delta\left(f_{X1}+\frac{1}{P},f_{Y1}\right)\right]$$
$$\otimes D_1^2\,\mathrm{sinc}(D_1 f_{X1})\,\mathrm{sinc}(D_1 f_{Y1}).$$

- Then perform the convolution:

$$\Im\{U_1(x_1,y_1)\} = \frac{D_1^2}{2}\mathrm{sinc}(D_1 f_{Y1})$$
$$\times\left\{\mathrm{sinc}(D_1 f_{X1}) - \frac{1}{2}\mathrm{sinc}\left[D_1\left(f_{X1}+\frac{1}{P}\right)\right] - \frac{1}{2}\mathrm{sinc}\left[D_1\left(f_{X1}-\frac{1}{P}\right)\right]\right\}.$$

Gratings (Cosine Magnitude Example)

- Substitute $x_2/\lambda z = f_{X1}$ *and* $y_2/\lambda z = f_{Y1}$ and apply the multipliers to get the Fraunhofer field:

$$U_2(x_2, y_2) = \frac{\exp(jkz)}{j\lambda z}\exp\left[j\frac{k}{2z}\left(x_2^{\,2}+y_2^{\,2}\right)\right]\frac{D_1^{\,2}}{2}\operatorname{sinc}\left(\frac{D_1}{\lambda z}y_2\right)$$
$$\times\left\{\operatorname{sinc}\left(\frac{D_1}{\lambda z}x_2\right)-\frac{1}{2}\operatorname{sinc}\left[\frac{D_1}{\lambda z}\left(x_2+\frac{\lambda z}{P}\right)\right]-\frac{1}{2}\operatorname{sinc}\left[\frac{D_1}{\lambda z}\left(x_2-\frac{\lambda z}{P}\right)\right]\right\}. \quad (9)$$

- The irradiance is the squared magnitude of Eq. (9).
- The following script portion evaluates the analytic irradiance result:

```
%analytic
[X2,Y2]=meshgrid(x2,y2);
lz=lambda*z;
u2a=(1/lz)*D1^2/2*sinc(D1/lz*Y2)...
     .*(sinc(D1/lz*X2)-1/2*sinc(D1/lz*(X2+lz/P))...
    -1/2*sinc(D1/lz*(X2-lz/P)));
I2a=abs(u2a).^2;
```

Gratings (Cosine Magnitude Example)

- **The front complex exponential terms for the Fraunhofer pattern were not coded up since only the irradiance is being examined—don't forget $1/\lambda z$.**
- Try this and see if the discrete and analytic plots come out the same.
- **The main application for a grating is wavelength separation.**

Gratings (Cosine Magnitude Example)

- Figure 10 shows superimposed curves for λ= 0.5μm and λ= 0.6μm.
- <u>**The first-order peaks are clearly separated.**</u>

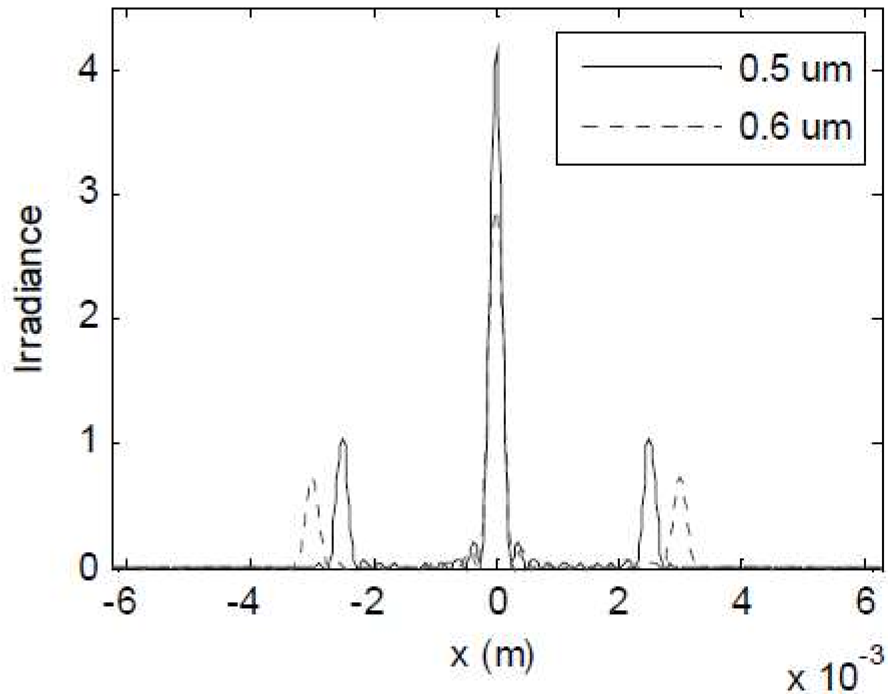


Fig. 10. Irradiance profiles for grating_cos.