Optical Signal Processing

Prof. Inkyu Moon

Dept. of Robotics Engineering, DGIST

Chapter 2
Scalar Diffraction and Propagation Solutions

Monochromatic Fields and Irradiance

- A monochromatic (single-frequency) scalar field propagating in free space can be expressed as

$$u(P,t) = A(P)\cos\left[2\pi \nu t - \varphi(P)\right], \qquad (1)$$

- where $A(P)$ is the amplitude and $\varphi(P)$ is the phase at a position P in space (x, y, z coordinates) and v is the temporal frequency.
- This expression models a propagating transverse optical (electric) field of a single polarization.
- **Monochromatic light provides the basis for analysis of diffraction theory.**
- A truly monochromatic light source is coherent.

Monochromatic Fields and Irradiance

- To give an example, a specific form of Eq. (1) corresponding to a plane wave propagating in the z direction would be

$$u(z,t) = A\cos[2\pi\nu t - kz],$$

- where the wavenumber k is defined as

$$k = \frac{2\pi}{\lambda},$$

- where λ is the vacuum wavelength.
- $\nu = c/\lambda$, **where c is the speed of light in vacuum**.
- The speed c can be derived from the argument (or phase) in the cos function.
- This wave has no dependence on x and y and is treated as extending infinitely in these directions.

Monochromatic Fields and Irradiance

- If the field in Eq. (1) is propagating in a linear medium (assumed for scalar diffraction), **the temporal frequency of the resulting field will remain unchanged**: we don't need to consider the temporal term.
- Furthermore, substituting a complex exponential (phasor) form for the cosine function provides a valid propagation result and aids in mathematical manipulation.

Monochromatic Fields and Irradiance

- These changes lead to a function that simply describes the spatial distribution of the field

$$U(P) = A(P)\exp[j\phi(P)]. \qquad (2)$$

- To further refine Eq. (2), the dependence on the *z* position can be removed, **where *z* is assumed to be the fundamental propagation direction**.
- $U_1(x,y) = A_1(x,y)\exp[j\phi_1(x,y)].$ indicates the field in the *x*–*y* plane is located at some position "1" on *z* axis.

Monochromatic Fields and Irradiance

- Detectors cannot follow the extremely high-frequency oscillations **(>10^{14} Hz: temporal frequency)** of the optical electric field.
- **Instead, optical detectors respond to the time-averaged squared magnitude of the field.**
- A quantity of considerable interest is the *irradiance*, which is defined here as (refer to *Poynting* theorem (see Schaum's Outline Optics))

$$I_1(x,y) = U_1(x,y)U_1(x,y)^* = |U_1(x,y)|^2 . \qquad (3)$$

- Expression (3) actually represents a shortcut for determining the time-averaged square magnitude of the field and is valid when the field is modeled by a complex phasor.

Optical Path Length and Field Phase Representation

- The refractive index *n* of a medium is the ratio of the speed of light in vacuum to the speed in the medium: for example, a typical glass used for visible light might have an index of about 1.6.
- For light propagating a distance *d* in a medium of index *n*, the **optical path length** (OPL) is defined:

$$\mathrm{OPL} = nd \; .$$

Optical Path Length and Field Phase Representation

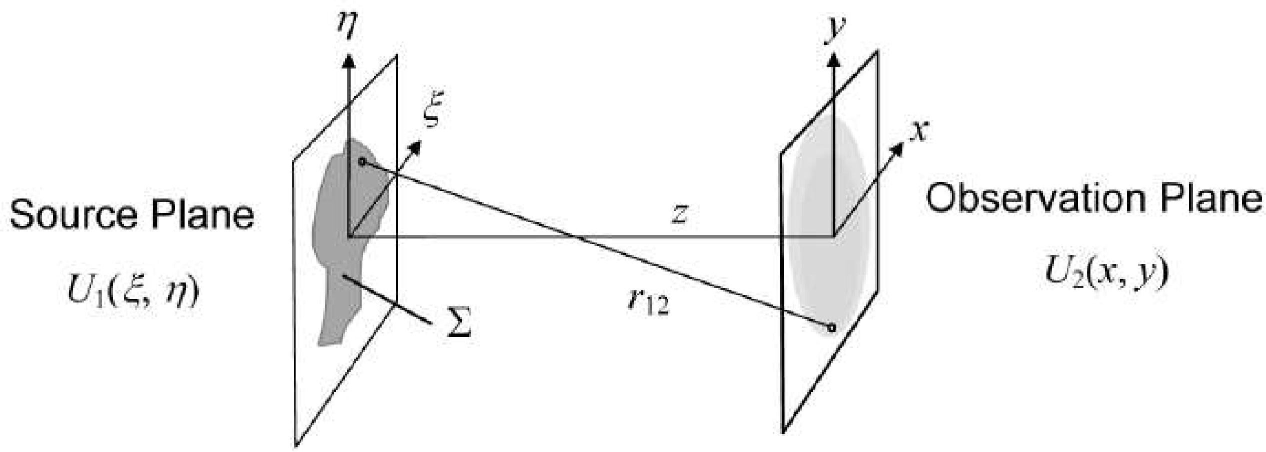
- The OPL multiplied by the wavenumber *k* shows up in the phase of the complex exponential used to model the optical field.
- If the plane wave propagates a distance *d* through a piece of glass with index *n*, then the field phasor representation is

$$U(d) = A\exp(jknd).$$

- **<u>The wavelength shortens to *λ/n* in the glass</u>** (temporal frequency remain same).

Analytic Diffraction Solutions

- Consider the propagation of monochromatic light from a 2D plane (source plane) indicated by the coordinate variables ξ and η.

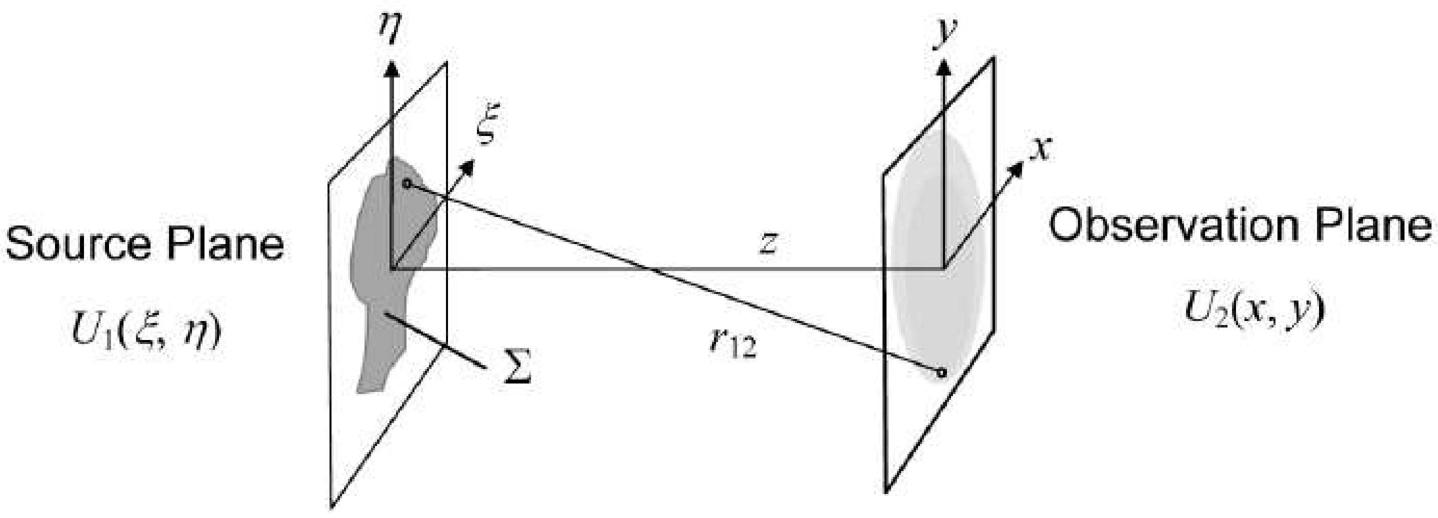


- **<u>At the source plane, an area Σ defines the extent of a source or an illuminated aperture</u>.**

Analytic Diffraction Solutions

- **The field distribution in the source plane is given by $U_1(\xi, \eta)$, and the field $U_2(x, y)$ in a distant observation plane can be predicted using the *Rayleigh–Sommerfeld diffraction solution.***

$$U_2(x,y) = \frac{z}{j\lambda} \iint_{\Sigma} U_1(\xi,\eta) \frac{\exp(jkr_{12})}{r_{12}^2} d\xi d\eta. \quad (4)$$



Analytic Diffraction Solutions

- λ is the optical wavelength, *k* is the wavenumber, which is equal to $2\pi/\lambda$ for free space, *z* is the distance between the centers of the source and observation coordinate systems and r_{12} is the distance between a position on the source plane and a position in the observation plane.

$$U_2(x,y) = \frac{z}{j\lambda} \iint_{\Sigma} U_1(\xi,\eta) \frac{\exp(jkr_{12})}{r_{12}^2} d\xi d\eta. \tag{4}$$

- ξ and η are variables of integration, and the integral limits correspond to the area of the source Σ.

- With the source and observation positions defined on parallel planes, the distance r_{12} is

$$r_{12} = \sqrt{z^2 + (x-\xi)^2 + (y-\eta)^2}\ . \tag{4-1}$$

Analytic Diffraction Solutions

- Expression (4) is a statement of the **Huygens–Fresnel principle**.
- This principle supposes the source acts as an infinite collection of fictitious point sources, each producing a spherical wave associated with the actual source field at any position (ξ, η).
- **The contributions of these spherical waves are summed at the observation position (x, y), allowing for interference**.

Analytic Diffraction Solutions

- Expression (4) is, in general, a superposition integral, but with the source and observation areas defined on parallel planes, **it becomes a convolution integral, which can be written as**

$$U_2(x,y) = \iint U_1(\xi,\eta)h(x-\xi, y-\eta)\,d\xi\,d\eta\,. \qquad (5)$$

- where the general form of the Rayleigh–Sommerfeld impulse response is

$$h(x,y) = \frac{z}{j\lambda}\frac{\exp(jkr)}{r^2}, \quad r = \sqrt{z^2 + x^2 + y^2}\,.$$

Analytic Diffraction Solutions

- The Fourier convolution theorem is applied to write Eq. (5) as

$$U_2(x,y) = \Im^{-1}\{\Im\{U_1(x,y)\}\Im\{h(x,y)\}\}. \quad (6)$$

- An equivalent expression for Eq. (6) is

$$U_2(x,y) = \Im^{-1}\{\Im\{U_1(x,y)\}H(f_X,f_Y)\},$$

- **where *H* is the Rayleigh–Sommerfeld transfer function given by**

$$H(f_X,f_Y) = \exp\left(jkz\sqrt{1-(\lambda f_X)^2-(\lambda f_Y)^2}\right). \quad (7)$$

- Strictly speaking, $\sqrt{f_X^2+f_Y^2} < 1/\lambda$ must be satisfied for propagating field components.

Analytic Diffraction Solutions

- The Rayleigh–Sommerfeld expression is the most accurate diffraction solution.
- Other than the assumption of scalar diffraction, **this solution only requires that $r \gg \lambda$ (refer to green theory), the distance between the source and the observation position, be much greater than a wavelength.**

Fresnel Approximation

- The square root in the distance terms can make analytic manipulations of the Rayleigh–Sommerfeld solution difficult and add execution time to a computational simulation.
- **By approximations for these terms, a more convenient scalar diffraction form is obtained.**
- Consider the binomial expansion

$$\sqrt{1+b} = 1 + \frac{1}{2}b - \frac{1}{8}b^2 + \ldots, \quad (8)$$

- where *b* is a number less than 1, then expand Eq. (4-1) and keep the first two terms to yield

$$r_{12} \approx z\left[1 + \frac{1}{2}\left(\frac{x-\xi}{z}\right)^2 + \frac{1}{2}\left(\frac{y-\eta}{z}\right)^2\right].$$

Fresnel Approximation

- This approximation is applied to the distance term in the phase of the exponential in Eq. (4), which amounts to assuming a parabolic radiation wave rather than a spherical wave.
- **Furthermore, use the approximation $r_{12} \approx z$ in the denominator of Eq. (4) to arrive at the *Fresnel diffraction* expression:**

$$U_2(x,y) = \frac{e^{jkz}}{j\lambda z} \iint U_1(\xi,\eta) \exp\left\{ j \frac{k}{2z} \left[(x-\xi)^2 + (y-\eta)^2 \right] \right\} d\xi d\eta \, .$$

Fresnel Approximation

- This expression is also a convolution, where the impulse response is

$$h(x,y) = \frac{e^{jkz}}{j\lambda z} \exp\left[\frac{jk}{2z}\left(x^2 + y^2\right)\right],$$

- and the transfer function is

$$H(f_X, f_Y) = e^{jkz} \exp\left[j\pi\lambda z\left(f_X^2 + f_Y^2\right)\right].$$

Fresnel Approximation

- Another useful form of the Fresnel diffraction expression is obtained by moving the quadratic phase term that is a function of *x* and *y* outside the integrals:

$$U_2(x,y) = \frac{\exp(jkz)}{j\lambda z}\exp\left[j\frac{k}{2z}\left(x^2+y^2\right)\right] \times \iint\left\{U_1(\xi,\eta)\exp\left[j\frac{k}{2z}\left(\xi^2+\eta^2\right)\right]\right\}\exp\left[-j\frac{2\pi}{\lambda z}(x\xi+y\eta)\right]d\xi d\eta. \qquad (9)$$

- Along with the amplitude and chirp multiplicative factors out front, **this expression is a Fourier transform of the source field times a chirp function where the following frequency variable substitutions are used for the transform:**

$$f_\xi \to \frac{x}{\lambda z}, \qquad f_\eta \to \frac{y}{\lambda z}.$$

Fraunhofer Approximation

- **Fraunhofer diffraction (diffraction patterns in "far field") can be obtained mathematically by approximating the chirp term multiplying the initial field within the integrals of Eq. (9) as unity**.
- The assumption involved is

$$z >> \left(\frac{k(\xi^2 + \eta^2)}{2} \right)_{max}, \tag{10}$$

- The *Fraunhofer diffraction* expression

$$U_2(x,y) = \frac{\exp(jkz)}{j\lambda z} \exp\left[j\frac{k}{2z}(x^2+y^2) \right] \times \iint U_1(\xi,\eta) \exp\left[-j\frac{2\pi}{\lambda z}(x\xi + y\eta) \right] d\xi d\eta. \tag{11}$$

Fraunhofer Approximation

- The condition of Eq. (10) requires very long propagation distances relative to the source support size.
- **Form of the Fraunhofer pattern also appears in the propagation analysis involving lenses**.
- The Fraunhofer diffraction expression is a powerful tool for many applications such as laser beam propagation, image analysis, and microscopy.
- **The Fraunhofer expression can be recognized simply as a Fourier transform of the source field with the variable substitutions**:

$$f_\xi \rightarrow \frac{x}{\lambda z}, \qquad f_\eta \rightarrow \frac{y}{\lambda z} \quad . \qquad (11\text{-}1)$$

Fraunhofer Approximation

- The Fraunhofer expression cannot be written as a convolution integral (no impulse response or transfer function).
- **<u>Scaled version of the Fourier transform of the initial field.</u>**

Fraunhofer Diffraction Example

- Consider a circular aperture illuminated by a unit amplitude plane wave.
- The complex field immediately beyond the aperture plane is

$$U_1(\xi,\eta) = \mathrm{circ}\left(\frac{\sqrt{\xi^2+\eta^2}}{w}\right). \qquad (12)$$

- To find the Fraunhofer diffraction field, the Fourier transform is taken as

$$\Im\{U_1(\xi,\eta)\} = w^2 \frac{J_1\left(2\pi w\sqrt{f_\xi^{\,2}+f_\eta^{\,2}}\right)}{w\sqrt{f_\xi^{\,2}+f_\eta^{\,2}}}.$$

Fraunhofer Diffraction Example

- Using Eq. (11), the field is expressed by:

$$U_2(x,y) = \frac{\exp(jkz)}{j\lambda z} \exp\left(j \frac{k}{2z} \left(x^2 + y^2 \right) \right) \times w^2 \frac{J_1\left(2\pi \frac{w}{\lambda z} \sqrt{x^2 + y^2} \right)}{\frac{w}{\lambda z} \sqrt{x^2 + y^2}}. \qquad (13)$$

- The irradiance is

$$I_2(x,y) = \left(\frac{w^2}{\lambda z} \right)^2 \left[\frac{J_1\left(2\pi \frac{w}{\lambda z} \sqrt{x^2 + y^2} \right)}{\frac{w}{\lambda z} \sqrt{x^2 + y^2}} \right]^2 .$$

Fraunhofer Diffraction Example

- Let's exercise MATLAB to display this irradiance pattern.
- Suppose w = 1mm, λ = 0.633μm (He–Ne laser wavelength), z = 50 m, and L = 0.2 m.

```
%fraun_circ - Fraunhofer irradiance plot

L=0.2;              %side length (m)
M=250;              %# samples
dx=L/M;             %sample interval
x=-L/2:dx:L/2-dx; y=x;   %coords
[X,Y]=meshgrid(x,y);

w=1e-3;             %x half-width
lambda=0.633e-6;%wavelength
z=50;               %prop distance
k=2*pi/lambda;      %wavenumber
lz=lambda*z;

%irradiance
I2=(w^2/lz)^2.*(jinc(w/lz*sqrt(X.^2+Y.^2))).^2;
```

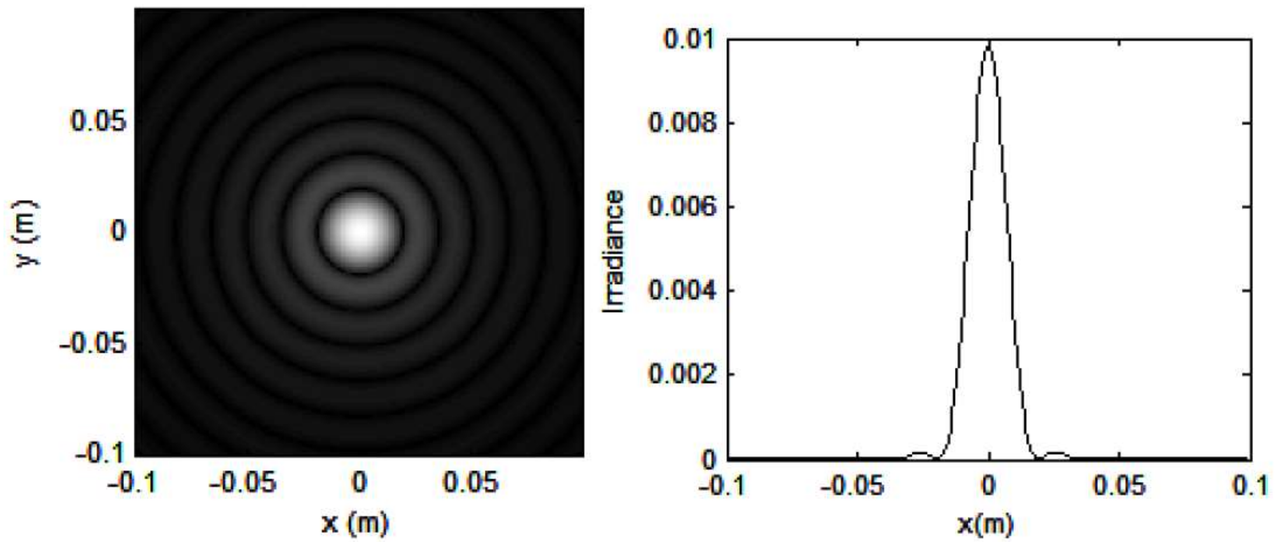
For jinc(), see the page 28.

Fraunhofer Diffraction Example

```
figure(1)    %irradiance image
imagesc(x,y,nthroot(I2,3));
xlabel('x  (m)'); ylabel('y  (m)');
colormap('gray');
axis square;
axis xy;
```

```

figure(2)    %x-axis profile
plot(x,I2(M/2+1,:));
xlabel('x(m)'); ylabel('Irradiance');
```



(a) (b)

- **Fig. 1** Fraunhofer irradiance (a) image pattern and (b) *x*-axis profile for a circular aperture.
- <u>**This is known as the Airy pattern.**</u>

Fraunhofer Diffraction Example

- It is helpful to make a function that handles the Bessel function ratio.
- In a New M-file (named “jinc”) enter the following:

```
function[out]=jinc(x);
%
% jinc function
%
% J1(2*pi*x)/x
% divide by zero fix
%
% locate non-zero elements of x
mask=(x~=0);
% initialize output with pi (value for x=0)
out=pi*ones(size(x));
% compute output values for all other x
out(mask)=besselj(1,2*pi*x(mask))./(x(mask));
end
```