Optical Signal Processing

Introduction

What is purpose?

- This course provides students with a basic understanding of the scientific principles associated with 1) Fourier optics, 2) image capture and formation, and 3) intelligent holographic **CONCITED FOR SET ASSEM SCHOUTS AND AREA SET ASSEM SCHOOLS AND AREA SET AND AREA SET AND AREA SET AND AREA SET APPLICATIONS.**
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- Digital holographic microscopy (DHM) is also introduced in this course for three-dimensional (3D) and quantitative sensing, imaging and measuring of biological and microscopic samples.

What are the prerequisites?

Fourier analysis & Basic optics

Introduction

Why are you in the class?

- Foundation for most 3D imaging system & modeling of a digital holographic microscope
- For MS or Ph.D. examinations.

Grading

- Midterm Exam: 40%
- Final Exams: 50%
- Homework/Class Participation: 10%

Reference

• J. Goodman, Introduction to Fourier Optics, Mcgraw-Hill, USA 1996

Chapter 1 Fourier Theory Review

A Little History and Purpose

- The study of **Optical Signal Processing** today leads naturally toward the computer for the following reason: Fast Fourier transform (FFT) algorithm provides an extremely efficient computational approach for solving wave optics problems.
- The FFT's speed makes it possible to perform thousands of optical propagation or imaging simulations in a reasonable amount of time.
- The methods explored in this course form the basis for wave (or physical) optics simulation tools that are widely used in industry.

A Little History and Purpose

- This course also provides step-by-step instructions for coding Fourier optics with MATLAB software.
- The end of this course, you can program basic Fourier optics problems—at least that's the goal! • This course also provides step-by-step instructions
for coding Fourier optics with MATLAB software.

• The end of this course, you can program basic

Fourier optics problems—at least that's the goal!

• lencourage you to
- I encourage you to consult some references for basic Fourier theory.
- Digital Signal Processing

Linear Systems and Nonlinear Systems **tems and Nonlinear**
 e Function

mpulse function $\partial(t)$ plays a central

analysis: It has the following prope
 $(t)=\begin{cases} 0 & t\neq 0 \\ \infty & t=0 \end{cases}$ $\int_{-\varepsilon}^{\varepsilon} \delta(t) dt = 1$

cannot be an ordinary function and

tically it is

Unit Impulse Function

The unit impulse function $\partial(t)$ plays a central role in system analysis: It has the following properties: **nd Nonlinear

and Nonlinear

and** $\delta(t)$ **plays a central role

t has the following properties:
** $\begin{array}{r} \epsilon^0 \to \epsilon^0 \to \epsilon^0 \end{array}$ **
** $\begin{array}{r} \epsilon^0 \to \epsilon^0 \to \epsilon^0 \end{array}$ **
** $\begin{array}{r} \epsilon^0 \to \epsilon^0 \to \epsilon^0 \end{array}$ **
** ϵ^1 **an ordinary function a** tion
 $\begin{array}{ll}\n\text{From} \\
\text{function} & \delta(t) \text{ plays a central role} \\
\text{is: It has the following properties:} \\
\frac{t \neq 0}{t = 0} & \int_{-\epsilon}^{\epsilon} \delta(t) dt = 1 \\
\text{It be an ordinary function and} \\
\text{it is defined by} \\
\phi(t) \delta(t) dt = \phi(0) \\
\gamma \text{ function continuous at } t = 0.\n\end{array}$

$$
\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} \qquad \int_{-\varepsilon}^{\varepsilon} \delta(t) dt = 1
$$

• Thus, $\delta(t)$ cannot be an ordinary function and mathematically it is defined by

$$
\int_{-\infty}^{\infty} \phi(t) \delta(t) dt = \phi(0)
$$

- where $\phi(t)$ is any function continuous at $t = 0$.
- Similarly, the delayed delta function $\delta(t-t_0)$ is defined by $\int_{0}^{\infty} \phi(t) \delta(t-t) dt$ $\int_{-\infty}^{\infty} \phi(t) \delta(t-t_0) dt = \phi$

Linear Systems and Nonlinear Systems

Unit Impulse Sequence (*n* is an integer)

The unit impulse (or unit sample) sequence δ [n] is defined: **15 and Nonlinear

Henry (1988)

1998 (or unit sample) sequence** ∂ **[n] i
** $[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$ **

shifted unit impulse (or sample)** \vert $n=0$

$$
\delta[n] = \begin{cases} \frac{n}{n+1} & n \neq 0 \\ 0 & n \neq 0 \end{cases}
$$

• Similarly, the shifted unit impulse (or sample) sequence δ [n-k] is defined as

Linear Systems and Nonlinear Systems

• If the operator T satisfies the following two conditions, then T is called a linear operator and the system represented by a linear operator T is called a linear system:

1. Additivity

• Given that $T\{x_1\} = y_1$ and $T\{x_2\} = y_2$ then $T\{x_1 + x_2\} = y_1 + y_2$ for any signals x_1 and x_2 .

2. Homogeneity

- $T\{\alpha x\} = \alpha y$ for any signals x and any scalar α .
- Two equations $T\{x_1+x_2\}=y_1+y_2$ and $T\{\alpha x\}=\alpha y$ can be combined into a single condition as

 $\mathbf{T} \{ \alpha_1 x_1 + \alpha_2 x_2 \} = \alpha_1 y_1 + \alpha_2 y_2$

Time-Invariant and Time-Varying Systems

- A system is called time-invariant if a time shift (delay or advance) in the input signal causes the same time shift in the output signal.
- For a continuous-time system, the system is timeinvariant if $\mathbf{T}\{x(t-\tau)\} = y(t-\tau)$ for any real value of τ .
- For a discrete-time system, the system is timeinvariant or shift-invariant if $\mathbf{T}\{x[n-k]\} = y[n-k]$ for any integer k.
- A system which does not satisfy the above Equations is called a time-varying system.

Linear Time-Invariant Systems

- If the system is linear and time-invariant, then it is called a linear time-invariant (LTI) system.
- The input-output relationship for LTI systems is described in terms of a convolution operation.
- The importance of the convolution operation in LTI systems stems from the fact that knowledge of the response of an LTI system to the unit impulse input allows us to find its output to any input signals.

Response of Continuous-Time LTI System

A. Impulse Response

The impulse response $h(t)$ of a continuous-time LTI system (represented by T) is defined to be the response of the system when the input is $\delta(t)$: **onse**

esponse $h(t)$ of a continuous-time LTI

sented by T) is defined to be the

ne system when the input is $\partial(t)$:
 $h(t) = T\{\delta(t)\}$
 **n Arbitrary Input

can be expressed as**
 $x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau$

em is linea

 $h(t) = \mathbf{T} \{\delta(t)\}\$

B. Response to an Arbitrary Input

The input $x(t)$ can be expressed as

 ∞ $=\int_{-\infty}^{\infty}x(\tau)\delta(t-\tau)$

• Since the system is linear, the response $y(t)$ of the system to the input $x(t)$ can be expressed as

$$
u(t) = \int_{t_0}^{t_0} f(t) dt
$$
\n
$$
= \int_{t_0}^{t_0} f(t) dt
$$
\n<math display="</math>

Response of Continuous-Time LTI System **Continuous-Time LTI**

tem is time-invariant, we have
 $h(t-\tau) = T\{\delta(t-\tau)\}$ (2)

Eq. (2) into Eq. (1), we obtain
 $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$ (3)

indicates that a continuous-time LTI

inpletely characterized by its impuls

- Since the system is time-invariant, we have $h(t-\tau) = \mathbf{T} \{\delta(t-\tau)\}$ (2)
- Substituting Eq. (2) into Eq. (1), we obtain

$$
y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau
$$
 (3)

• Equation (3) indicates that a continuous-time LTI system is completely characterized by its impulse response h(t).

Response of Continuous-Time LTI System **of Continuous-Time LTI**

(3) defines the convolution of two

us-time signals $x(t)$ and $h(t)$ denoted by
 $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$ (4)

(4) is called the *convolution integral*.

• Equation (3) defines the convolution of two continuous-time signals $x(t)$ and $h(t)$ denoted by

$$
y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau
$$
 (4)

- Equation (4) is called the *convolution integral*.
- **The output of any continuous-time LTI system is** the convolution of the input $x(t)$ with the impulse response $h(t)$ of the system.

Response of a Discrete-Time LTI System

A. Impulse Response

• The impulse response (or unit sample response) $h[n]$ of a discrete-time LTI system (represented by T) is defined to be the response of the system when the input is $\delta[n]$, that is, **Solution**

sponse (or unit sample response)

te-time LTI system (represented by

be the response of the system

t is $\delta[n]$, that is,
 $h[n] = T\{\delta[n]\}$
 **a Arbitrary Input

<u>can be expressed as</u>
** $[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

$$
h[n] = \mathbf{T}\{\delta[n]\}
$$

B. Response to an Arbitrary Input

- The input x[n] can be expressed as k $x[n] = \sum_{k=1}^{n} x[k]\delta[n-k]$ $\overline{\infty}$ $=-\infty$ $= \sum_{n=1}^{\infty} x[k] \delta[n-k]$ (5)
- Since the system is linear, the response $y[n]$ of the system to an input $x[n]$ can be expressed as:

Let
$$
S[n]
$$
, $S[n]$, $S[n]$, $S[n]$, $S[n]$, $S[n]$, $S[n]$.

\n**to an Arbitrary Input**

\n**to an Arbitrary Input**

\n**to (a) A**

\n**to (b) A**

\n**to (c) A**

\n**to (d) A**

\n**to (e) A**

\n**to (f) B**

\n**to (g) A**

\n**to (h) B**

\n**to (i) A**

\n**to (j) A**

\n**to (k) B**

\n**to (l) A**

Response of a Discrete-Time LTI System

- Since the system is time-invariant, we have $h[n-k] = \mathbf{T} \{\delta[n-k] \}$ (7) **a Discrete-Time LTI**

em is time-invariant, we have
 $h[n-k] = T\{\delta[n-k]\}$ (7)

q. (7) into Eq. (6), we obtain
 $[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$ (8)

mdicates that a discrete-time LTI

mlately share texing by its impulse
- Substituting Eq. (7) into Eq. (6), we obtain

$$
y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]
$$
 (8)

• Equation (8) indicates that a discrete-time LTI system is completely characterized by its impulse response h[n].

Convolution Sum

• Equation (8) defines the convolution of two sequences $x[n]$ and $h[n]$ denoted by

On Sum

\n(8) defines the convolution of two

\nes
$$
x[n]
$$
 and $h[n]$ denoted by

\n
$$
y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]
$$

\n(9) is called the convolution sum.

- Equation (9) is called the convolution sum.
- The output of any discrete-time LTI system is the convolution of the input $x[n]$ with the impulse response $h[n]$ of the system.

Frequency Response of Continuous-Time LTI Systems

• We know that the output $y(t)$ of a continuous-time LTI system equals the convolution of the input $x(t)$ with the impulse response $h(t)$: that is,

 $y(t) = x(t) * h(t)$

• Applying the **convolution property**, we obtain

 $Y(\omega) = X(\omega)H(\omega)$

- where $Y(\omega)$, $X(\omega)$, and $H(\omega)$ are the Fourier transforms of $y(t)$, $x(t)$, and $h(t)$, respectively.
- $H(\omega)$ is called the system's frequency response.

Fourier Analysis of Signals

- The Fourier representation of signals plays an extremely important role in both continuous-time and discrete-time signal processing.
- We review the continuous-time Fourier transform (FT) and the FT for discrete-time signals).
- Review Fourier series and Fourier transform which convert time-domain signals into frequencydomain (or spectral) representations.

Complex Exponential Signals

- The complex exponential signal $x(t) = e^{iw_0 t}$ is an important example of a complex signal.
- The fundamental period T_0 of $x(t)$ is given by: 0 $\overline{2}$ $T_0=\frac{2\pi}{\pi}$ $\omega_{\scriptscriptstyle\text{o}}$
- Any signals can be expressed by using the complex exponential form. $\overline{0}$
- The complex exponential sequence is of the form.

$$
x[n] = e^{j\Omega_0 n}
$$

Any sequences can be expressed by using the complex exponential form.

Fourier Analysis of Continuous-Time **Signals**

Periodic Signals

• We define a continuous-time signal $x(t)$ to be periodic if there is a positive nonzero value of *for* which

 $x(t+T) = x(t)$ all t

• Two basic examples of periodic signals are the real sinusoidal signal

 $x(t) = \cos(\omega_0 t + \phi)$

• and the complex exponential signal

$$
x(t) = e^{j\omega_0 t}
$$

• where $\omega_0 = 2\pi/T_0 = 2\pi f_0$ is called the **fundamental** angular frequency

Fourier Analysis of Continuous-Time **Signals ilysis of Continuous-Tim

inential Fourier Series Representation

<u>ex exponential Fourier series</u>

tion of a periodic signal x(t) with

<u>ial period T_o is given by</u>
** $(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_k t}$ $\omega_0 = \frac{2\pi}{T_0}$ **

re**

Complex Exponential Fourier Series Representation

The complex exponential Fourier series representation of a periodic signal $x(t)$ with <u>fundamental period T_o is given by</u> <u>rier Series Representation
tial Fourier series</u>
riodic signal x(t) with
pis given by
 $\omega_0 = \frac{2\pi}{T_0}$
the complex Fourier
ven by
(t)e^{-jka₀t} dt

$$
x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}
$$

• where c_k are known as the **complex Fourier** coefficients and are given by

$$
c_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt
$$

Fourier Analysis of Continuous-Time **Signals**

Trigonometric Fourier Series

The trigonometric Fourier series representation of a periodic signal $x(t)$ with fundamental period $\overline{\mathcal{I}}_{\mathbf{Q}}$, is given by: **Analysis of Continuous-Time**
 Solution
 Interference Series
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<u>urier series representation</u>

don't by the fundamental period

for** $\omega_0 t + b_k \sin k \omega_0 t$ **)** $\omega_0 = \frac{2\pi}{T_0}$ **

rier coefficients given by:

(t) cos k** $\omega_0 t dt$ **

(t) sin k** $\omega_0 t dt$ **urier series representation

t) with fundamental period
** $\omega_0 t + b_k \sin k \omega_0 t$ $\omega_0 = \frac{2\pi}{T_0}$ **

rier coefficients given by:

(***t***) cos** $k\omega_0 t dt$ **

(***t***) sin** $k\omega_0 t dt$

$$
x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos k\omega_0 t + b_k \sin k\omega_0 t \right) \qquad \omega_0 = \frac{2\pi}{T_0}
$$

• a_k , and b_k , are the Fourier coefficients given by:

$$
a_k = \frac{2}{T_0} \int_{T_0} x(t) \cos k\omega_0 t dt
$$

$$
b_k = \frac{2}{T_0} \int_{T_0} x(t) \sin k\omega_0 t dt
$$

Fourier Analysis of Continuous-Time **Signals Continuous-Time**
 dic Signal
 a periodic signal x(t) over
 $(t)^2 dt$

 the complex exponential

Power Content of a Periodic Signal

The average power of a periodic signal $x(t)$ over any period is given by: Periodic Signal

Periodic signal x(t) over

<u>n by</u>:
 $\frac{1}{r_o}\int_{\tau_o} |x(t)|^2 dt$

ed by the complex exponential

in it can be shown that
 $(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$

$$
P = \frac{1}{T_0} \int_{T_0} \left| x(t) \right|^2 dt
$$

• If $x(t)$ is represented by the complex exponential Fourier series, then it can be shown that

$$
\frac{1}{T_0}\int_{T_0}\left|x(t)\right|^2 dt = \sum_{k=-\infty}^{\infty}\left|c_k\right|^2
$$

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 gnals

From Fourier Series to Fourier Transform

• Let $x(t)$ be a nonperiodic signal of finite duration:
 $x(t) = 0$ $|x| > T$ Fourier Analysis of Continuous-Time **Signals**

From Fourier Series to Fourier Transform

• Let $x_{T_0}(t)$ be a periodic signal formed by repeating $\overline{}$

• If we let $T_0 \rightarrow \infty$, we have $\lim_{T_0 \rightarrow \infty} x_{T_0}(t) = x(t)$ $\lim_{T_0 \to \infty} x_{T_0}$ $x_{T_0}(t) = x(t)$ $\rightarrow \infty$ $=$

Fourier Analysis of Continuous-Time **Signals ysis of Continuous-Time**

exponential Fourier series of $x_{T_0}(t)$ i
 $(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_k t}$ $\omega_0 = \frac{2\pi}{T_0}$ **f Continuous-Time**

mtial Fourier series of $x_{T_0}(t)$ is:
 $e^{jk\omega_0 t}$ $\omega_0 = \frac{2\pi}{T_0}$ (10)
 $\omega_0^{i/2}$
 $x_{T_0/2}x_{T_0}(t)e^{-jk\omega_0 t}dt$ (10a)

• The complex exponential Fourier series of $x_{T_0}(t)$ is:

$$
x_{T_0}\left(t\right) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \qquad \omega_0 = \frac{2\pi}{T_0} \qquad \qquad (10)
$$

• where

$$
(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}
$$
 (10)

$$
c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0} (t) e^{-jk\omega_0 t} dt
$$
 (10a)
(t) for $|t| < T_0/2$ and also since $x(t) = 0$
interval, Eq. (10a) can be rewritten as:

$$
\int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt
$$

• Since $x_{T_0}(t) = x(t)$ for $|t| < T_0/2$ and also since $x(t) = 0$ outside this interval, Eq. (10a) can be rewritten as:

$$
c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt
$$

Fourier Analysis of Continuous-Time **Signals ysis of Continuous-Time**
 $\begin{array}{c} x(\omega) \text{ as} \\ x(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \end{array}$ (11)

Fourier coefficients c_k , can be **Continuous-Time**
 $\begin{array}{|l|l|}\n\hline\n(10)e^{-j\omega t}dt & (11)\n\hline\n\end{array}$
 $\begin{array}{|l|l|}\n\hline\n\end{array}$
 $\begin{array}{|l|l|}\n\hline\n\end{array}$
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 $\begin{array}{|l|l|}\n\hline\n\end{array}$
 $\begin{array}{$

• Let us define $X(\omega)$ as

$$
X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt
$$
 (11)

• The complex Fourier coefficients c_k , can be expressed as

$$
c_k = \frac{1}{T_0} X(k\omega_0)
$$
 (11a)

• Substituting Eq. (11a) into Eq. (10), we have

ne
$$
X(\omega)
$$
 as
\n
$$
X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt
$$
\n(11)
\nex Fourier coefficients c_k , can be
\nas
\n
$$
c_k = \frac{1}{T_0} X(k\omega_0)
$$
\n(11a)
\n12b **ig Eq. (11a) into Eq. (10), we have**
\n
$$
x_{T_0}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} X(k\omega_0) e^{jk\omega_0 t}
$$
\n
$$
x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\omega_0) e^{jk\omega_0 t} \omega_0
$$
\n(12)

Fourier Analysis of Continuous-Time **Signals**

- As $T_0 \to \infty$, $\omega_0 = 2\pi/T_0$ becomes infinitesimal $(\omega_0 \to 0)$.
- Thus, $\omega_0 = \Delta \omega$ then Eq. (12) becomes

$$
x_{T_0}(t)|_{T_0\to\infty}\to\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}X(k\Delta\omega)e^{jk\Delta\omega t}\Delta\omega
$$
 (13)

• Therefore,

$$
x_{0} \rightarrow \infty, \omega_{0} = 2\pi/T_{0} \text{ becomes infinitesimal } (\omega_{0} \rightarrow 0).
$$

\n
$$
x_{0} \rightarrow \Delta \omega \text{ then Eq. (12) becomes}
$$

\n
$$
x_{0} (t)|_{T_{0} \rightarrow \infty} \rightarrow \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\Delta \omega) e^{jk\Delta \omega t} \Delta \omega
$$

\nTherefore,
\n
$$
x(t) = \lim_{T_{0} \rightarrow \infty} x_{T_{0}} (t) = \lim_{\Delta \omega \rightarrow 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\Delta \omega) e^{jk\Delta \omega t} \Delta \omega
$$
 (14)

Fourier Analysis of Continuous-Time **Signals**

- The sum on the right-hand side of Eq. (14) can be viewed as the area under the function $X(\omega)e^{j\omega t}$ as shown in the following figure. sis of Continuous-Time

e right-hand side of Eq. (14) can be

e right-hand side of Eq. (14) can be

e right-hand side of Eq. (14) can be

ollowing figure

obtain
 $(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$ (15)

representat
- Therefore, we obtain

$$
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega
$$
 (15)

It's the Fourier representation of a nonperiodic $x(t)$.

Fourier Analysis of Continuous-Time **Signals alysis of Continuous-Time**
 sform Pair
 $x(\omega)$ is called the Fourier transform

id Eq. (15) defines the inverse Fourier
 $\alpha(x) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$
 $t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega$

Fourier Transform Pair

The function $X(\omega)$ is called the Fourier transform of $x(t)$, and Eq. (15) defines the inverse Fourier transform of $X(\omega)$. **nsform Pair**
 nsform Pair

tion $X(\omega)$ is called the Fourier transform

nd Eq. (15) defines the inverse Fourier

m of $X(\omega)$.
 $X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
 $(x) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega$

$$
X(\omega) = \mathscr{F}\left\{x(t)\right\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt
$$

$$
x(t) = \mathscr{F}^{-1}\left\{X(\omega)\right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega
$$

• If impulse functions are permitted in the transform, some periodic signals (i.e., sin(x), cos(x)) can have Fourier transforms.

Fourier Analysis of Continuous-Time **Signals**

Fourier Spectra

The Fourier transform $X(\omega)$ of $x(t)$ is, in general, complex, and it can be expressed as:

 $X(\omega) = |X(\omega)|e^{j\phi(\omega)}$

- The quantity $|X(\omega)|$ is called the magnitude spectrum of $x(t)$, and $\phi(\omega)$ is called the phase spectrum of $x(t)$. **of x(t) is, in general,

essed as:**
 $\lim_{t \to \infty}$
 $\lim_{t \to \infty}$
 $\lim_{t \to \infty}$
 $X(-\omega) = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt$
- If $x(t)$ is a real signal, we get $X(-\omega) = \int_{-\infty}^{\infty}$ $-\omega$) = $\int_{-\infty}^{\infty}$
- Then it follows that

$$
X(-\omega) = X^*(\omega)
$$

$$
X(-\omega) = |X(\omega)| \qquad \phi(-\omega) = -\phi(\omega)
$$

• The amplitude spectrum $|X(\omega)|$ is an even function and the phase spectrum $\phi(\omega)$ is an odd one of ω .

Discrete Fourier Series

- A discrete-time signal (or sequence) $x[n]$ is periodic if there is a positive integer N for which $x[n+N] = x[n]$ all n
- The fundamental period N_0 of $x[n]$ is the smallest positive integer N.
- The complex exponential sequence

 $x[n] = e^{j(2\pi/N_0)n} = e^{j\Omega_0 n}$

• where $\Omega_0 = 2\pi/N_0$, is a periodic sequence with fundamental period N_{0} . .

Discrete Fourier Series

• One very important distinction between the discrete-time and the continuous-time complex exponential is that *the signals e^{j@ot}* are distinct for distinct values of ω_0 but the sequences $e^{j\Omega_0 n}$ which differ in frequency by a multiple of 2π , are identical. **Fourier Series**

ery important distinction between t

te-time and the continuous-time co

ential is that the signals $e^{j\omega_0 t}$ are dist

to values of ω_0 but the sequences

differ in frequency by a multiple of

cal.

• Let
$$
\Psi_k[n] = e^{jk\Omega_0 n}
$$
, $\Omega_0 = \frac{2\pi}{N_0}$, $k = 0, \pm 1, \pm 2,...$

- We have $\Psi_0[n] = \Psi_{N_0}[n]$ $\Psi_1[n] = \Psi_{N_0+1}[n]$... $\Psi_k[n] = \Psi_{N_0+k}[n]$...
- In case of $\hat{\Omega}_0 = \Omega_0 + 2\pi k$, $e^{j\hat{\Omega}_0 n} = e^{j\Omega_0 n} e^{j2\pi kn} = e^{j\Omega_0 n}$ because $e^{j2\pi kn}$ is always 1: note that k and n are integer.
- However, incase of $e^{jk\omega_0 t} = e^{j k \overline{T_0}^t} = e^{j \overline{T_0}/k^t}$, $k = 0, \pm 1, \pm 2, ..., e^{jk\omega_0 t}$ are differ: note that t is not integer. 2π , 2π $e^{jk\omega_0 t}=e^{jk\frac{2\pi}{T_0}t}=e^{j\frac{2\pi}{T_0/k}t},\quad k=0,\pm1,\pm2,...,\quad e^{jk\omega_0 t}$ π , 2π ${}^{\omega_0t}=e^{\int^{\kappa} \overline{T_0}^{\iota}}=e^{\int \overline{T_0/k}^{\iota}}, \quad k=0,\pm 1,\pm 2,...,\quad e^{jk\omega_0t}$

Discrete Fourier Series

The discrete Fourier series representation of a periodic sequence $x[n]$ with fundamental period $\overline{\mathsf{N}_{\mathsf{0}}}$ is given by **Irier Series**

Production of a

Production of a

<u>Internation of a</u>
 $[n] = \sum_{k=0}^{N_0-1} c_k e^{jk\Omega_0 n}$
 $\Omega_0 = \frac{2\pi}{N_0}$

the range of *k* is [0 N_0 -1] (see page 33) The state of the state of

$$
x[n] = \sum_{k=0}^{N_0 - 1} c_k e^{jk\Omega_0 n} \qquad \qquad \Omega_0 = \frac{2\pi}{N_0}
$$

Note that the range of k is [0 N_{0} -1] (see page 33)

where c_k are the Fourier coefficients and given by:

$$
c_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-jk\Omega_0 n}
$$

The DFS coefficients are periodic.

om Discrete Fourier Series to
 urier Transform (DTFT)

• Let $x[n]$ be a nonperiodic sequence of finite

duration: that is, for some positive integer N_1 ,
 $x[n] = 0$ $|n| > N_1$ From Discrete Fourier Series to Fourier Transform (DTFT)

duration: that is, for some positive integer N_1 , $\overline{}$

 $x[n] = 0$ $|n| > N_1$

• Such a sequence is shown in the following figure.

• Let $x_{N_0}[n]$ be a periodic sequence formed by repeating $x[n]$ with fundamental period N_0 as shown in the following figure.

From Discrete Fourier Series to Fourier Transform (DTFT)

- If we let $N_0 \to \infty$, we have $\lim_{N_0 \to \infty} x_{N_0}[n] = x[n].$
- The discrete Fourier series of $x_{N_a}[n]$ is given by

etc Fourier Series to
\n**nsform (DTFT)**
\n
$$
{}_{0} \rightarrow \infty
$$
, we have $\lim_{N_{0} \rightarrow \infty} x_{N_{0}}[n] = x[n]$.
\n**te Fourier series of** $x_{N_{0}}[n]$ **is given by**
\n
$$
x_{N_{0}}[n] = \sum_{k \in (N_{0})} c_{k} e^{j k \Omega_{0} n} \qquad \Omega_{0} = \frac{2\pi}{N_{0}}
$$
\n
$$
c_{k} = \frac{1}{N_{0}} \sum_{n \in (N_{0})} x_{N_{0}}[n] e^{-j k \Omega_{0} n}
$$
\n
$$
= x[n]
$$
 for $|n| \leq N_{1}$ and also since $x[n] = 0$
\nis interval, Eq. (17) can be rewritten as
\n
$$
x_{k} = \frac{1}{N_{0}} \sum_{n = -N_{1}}^{N_{1}} x[n] e^{-j k \Omega_{0} n} = \frac{1}{N_{0}} \sum_{n = -\infty}^{\infty} x[n] e^{-j k \Omega_{0} n}
$$

\n**ne** $X(\Omega)$ as
\n
$$
X(\Omega) = \sum_{n = -\infty}^{\infty} x[n] e^{-j \Omega n}
$$

\n**2,3,1, 0, -1, ...** and fundamental angular frequency in Ω
\n**2,5,1, 2,3, ...** N_{0} -1,...] and fundamental angular frequency in Ω
\n**2,6,1, 2, 3, ...**

• Since $x_{N_0}[n] = x[n]$ for $|n| \le N_1$ and also since $x[n] = 0$ outside this interval, Eq. (17) can be rewritten as

$$
c_k = \frac{1}{N_0} \sum_{n=-N_1}^{N_1} x[n] e^{-jk\Omega_0 n} = \frac{1}{N_0} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\Omega_0 n}
$$

• Let us define $X(\Omega)$ as

$$
X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}
$$

where n is [0,1,2,3,.. N_0 -1,...] and fundamental angular frequency in Ω is Ω_0 , exp[-j Ω n] \rightarrow exp[-j Ω_0 n]. So X(Ω) has 2 π period (see page 33).

From Discrete Fourier Series to Fourier Transform (DTFT)

- The Fourier coefficients c_k can be expressed as: (18) **Solution Series to Series to Series to Series (ASS)**
 DTFT)
 c_k can be expressed as:
 $(c_k \Omega_0)$ (18)
 Ω Eq. (16), we have 0 $c_k = \frac{1}{N} X (k \Omega)$ $N_{\rm c}$ $=\frac{1}{N}X(k\Omega_0)$ **Prourier Series to**
 form (DTFT)
 efficients c_k **can be expressed as:**
 $c_k = \frac{1}{N_o} X(k\Omega_o)$ (18)
 q. (18) into Eq. (16), we have
 $[n] = \frac{1}{2\pi} \sum_{k=(N_o)} X(k\Omega_o) e^{jk\Omega_o n} \Omega_o$ (19)
 c with period 2π and so is $e^{j\Omega$
- Substituting Eq. (18) into Eq. (16), we have (19) 0 0 $0e^{\theta}$ Ω_0 1 $\overline{2}$ $ik\Omega_0 n$ _C N_0 [$^{\prime \prime}$] $^{-}$ 2 π $\sum\limits_{k = \langle N_0 \rangle}$ $x_{N_0}\left[n\right]=\frac{1}{2\pi}\sum_{k=N\infty}X\left(k\Omega_0\right)e^{j k\Omega_0}$ $\overline{\Omega}_0$ $=$ $=\frac{1}{2\pi}\sum_{\alpha}X(k\Omega_{0})e^{ik\Omega_{0}n}\Omega_{0}$
- $X(\Omega)$ is periodic with period 2π and so is $e^{j\Omega n}$.
- So, the product $X(\Omega)e^{j\Omega n}$ is periodic with period 2 π .

where n is [0,1,2,3,.. N_0 -1,...] and fundamental angular frequency in Ω is Ω_0 , exp[-j Ω n] \rightarrow exp[-j Ω_0 n]. So X(Ω) has 2 π period (see page 33).

From Discrete Fourier Series to Fourier Transform (DTFT)

• As shown in the following figure, each term in the summation in Eq. (19) represents the area of a rectangle of height $X(k\Omega_0)e^{jk\Omega_0 n}$ and width Ω_0 .

**OM Discrete Fourier Series to

urier Transform (DTFT)**

• As $N_0 \rightarrow \infty$, $\Omega_0 = 2\pi/N_0$, becomes infinitesimal

($\Omega_0 \rightarrow 0$) and Eq. (19) passes to an integral.

• Furthermore, since the summation in Eq. (19) is From Discrete Fourier Series to Fourier Transform (DTFT)

- ($\Omega_{0} \rightarrow 0$) and Eq. (19) passes to an integral. • As $N_0 \rightarrow \infty$, $\Omega_0 = 2\pi / N_0$, becomes infinitesimal
- Furthermore, since the summation in Eq. (19) is over N_0 , consecutive intervals of width $\Omega_0 = 2\pi/N_0$, the total interval of integration will always have a width 2π . = $2\pi/N_0$, becomes infinitesimal
q. (19) passes to an integral.
since the summation in Eq. (19) is
ecutive intervals of width $\Omega_0 = 2\pi/N_0$,
val of integration will always have a
 ∞ Eq. (19) becomes
 $[n] = \frac{1}{2\pi} \int_{2$
- Thus, as $N_0 \rightarrow \infty$ Eq. (19) becomes

$$
x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega
$$
 (20)

From Discrete Fourier Series to Fourier Transform (DTFT)

• The function $X(\omega)$ is called the discrete-time Fourier transform (DTFT) of $x[n]$, and equation (20) defines the inverse Fourier transform of $X(\omega)$. **rete Fourier Series to**
 ansform (DTFT)
 sform Pair

tion $X(\omega)$ is called the discrete-time

ransform (DTFT) of $x[n]$, and equation (20)

the inverse Fourier transform of $X(\omega)$.

(Ω) = $\mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{\infty$ **arisform Pair**
 nsform Pair

tion $X(\omega)$ is called the discrete-time

ransform (DTFT) of $x[n]$, and equation (20)

the inverse Fourier transform of $X(\omega)$.
 $X(\Omega) = \mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$
 $[n] = \mathcal{F}^{-1}\{X(\$

$$
X(\Omega) = \mathscr{F}\left\{x[n]\right\} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}
$$

$$
x[n] = \mathscr{F}^{-1}\left\{X(\Omega)\right\} = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega
$$

**om Discrete Fourier Series to

urier Transform (DTFT)

Fourier Spectra

• The Fourier transform** $X(\omega)$ **of** $x[n]$ **is, in general,

complex and can be expressed as:
** $X(\Omega) = |X(\Omega)|e^{j\theta(\Omega)}$ From Discrete Fourier Series to Fourier Transform (DTFT)

Fourier Spectra

complex and can be expressed as:

 $X(\Omega) = |X(\Omega)|e^{j\phi(\Omega)}$

- The quantity $|X(\Omega)|$ is called the magnitude spectrum of $x[n]$, and $\phi(\Omega)$ is called the phase spectrum of $x[n]$.
- Furthermore, if $x[n]$ is real, the amplitude spectrum $|X(\Omega)|$ is an even function and the phase spectrum $\phi(\Omega)$ is an odd function.

Frequency Response of Discrete-Time LTI Systems

• The output $y[n]$ of a discrete-time LTI system equals the convolution of the input $x[n]$ with the impulse response $h[n]$; that is,

 $y[n] = x[n] * h[n]$

- Applying the convolution property, we obtain $Y(\Omega) = X(\Omega) H(\Omega)$
- where $Y(\Omega)$, $X(\Omega)$, and $H(\Omega)$ are the Fourier transforms of $y[n]$, $x[n]$, and $h[n]$, respectively.
- Unlike the frequency response of continuoustime systems, that of all discrete-time LTI systems is periodic with period 2π.

 $H(\Omega) = H(\Omega + 2\pi)$

-
- mpling

 Most discrete-time signals come from sampling a

<u>continuous-time signals come from sampling a</u>

<u>signals, radar, sonar data, and biological signals</u>. continuous-time signal, such as speech, audio signals, radar, sonar data, and biological signals.
- Sampling theory gives precise conditions under which an analog signal may be uniquely represented in terms of its samples.
- When implementing holography simulations on the computer we need to represent functions by sampled values and apply transform/processing methods designed for these discrete signals.
- It would be great to model the physical elements with many samples.

• Consider the two-dimensional (2D) analytic function $g(x,y)$ and suppose it is sampled in a uniform manner (see Fig. 2.1) in the x and y directions, which is indicated by

 $g(x, y) \rightarrow g(m\Delta x, n\Delta y).$

- Sample interval is Δx in the x direction and Δy in the y direction, and m and n are integer-valued indices of the samples: **the respective sample pling**

Sample interval is Δx in the x direction and Δy in

the y direction, and m and n are integer-valued

indices of the samples: **the respective sample**
 rates (sampling frequency) are 1/ Δx **and 1/** Δy .

I **rates (sample interval is** Δx **in the** *x* **direction and** Δy **in the** *y* **direction, and** *m* **and** *n* **are integer-valued indices of the samples: <u>the respective sample</u> rates (sampling frequency) are** $1/\Delta x$ **and** $1/\Delta y$ **.

I**
- **pling**

Sample interval is Δx in the x direction and Δy in

the y direction, and m and n are integer-valued

indices of the samples: **the respective sample**
 rates (sampling frequency) are 1/ Δx **and 1/** Δy .

I **Sample interval is** Δx **in the x direction and** Δy **in
the y direction, and m and n are integer-valued
indices of the samples: the respective sample
rates (sampling frequency) are** $1/\Delta x$ **and** $1/\Delta y$ **.
In practice, the sa** Sample interval is Δx in the *x* direction and Δy in
the *y* direction, and *m* and *n* are integer-valued
indices of the samples: **the respective sample**
rates (sampling frequency) are $1/\Delta x$ **and** $1/\Delta y$ **.**
In pract

$$
m = -\frac{M}{2}, \dots, \frac{M}{2} - 1, \qquad n = -\frac{N}{2}, \dots, \frac{N}{2} - 1.
$$

• This is a standard index arrangement where M and N are assumed to be even.

- A finite physical area spanned by the sampled space, and this is given by $L_X \times L_Y$, where L_X is the length along the x side of the sampled space and L_y is the length along the y side (see Fig. 2.1).
- They represent physical distances and are related to the sampling parameters by

 $L_x = M \Delta x$, $L_y = N \Delta y$.

• If $D_{\underline{X}}$ is the support in the x direction and $D_{\underline{Y}}$ is the support in the y direction, then for the significant values of $g(x,y)$ to be contained within the array **requires**

$$
D_{X} < L_{X}, \qquad D_{Y} < L_{Y}.
$$

• Another concern is whether the sample intervals are small enough to preserve features of $g(x,y)$.

Figure 2.2 Illustration of the (a) support D_x and (b) bandwidth B_x along the x axis of $g(x, y)$. Bandwidth is commonly defined as a half-width measure and is illustrated here with a profile of $|G(f_X, f_Y)|$, the Fourier transform magnitude of $g(x, y)$.

• If functions that are bandlimited, where the spectral content of the signal is limited to a finite range of frequencies, a continuous function can be recovered exactly from the samples if the sample interval is smaller than a specific value.

mpling
• The Shannon–Nyquist sampling theorem,
• Extended to two dimensions, states this
• Equirement as: extended to two dimensions, states this requirement as:

$$
\Delta x < \frac{1}{2B_x}, \qquad \Delta y < \frac{1}{2B_y} \tag{21}
$$

- where B_x is the bandwidth of the spectrum of the continuous function along the x direction and B_y is the bandwidth along the y direction.
- Bandwidth is illustrated in Fig. 2.2(b).
- Violating Eq. (21) results in aliasing, in which
undersampled high-frequency components in the $\Delta x < \frac{1}{2B_x}$, $\Delta y < \frac{1}{2B_y}$ (21)
where B_x is the bandwidth of the spectrum of the
continuous function along the x direction and B_y is
the bandwidth along the y direction.
Bandwidth is illustrated in Fig. 2.2(b).
 signal are interpreted erroneously as low frequency content.

Periodic Sampling

-
- Discrete-time signals are formed by periodically sampling a continuous-time signal

 $x(n) = x_a(nT_s)$

- The sample spacing T_s is the sampling period, and $f_{\underline{s}}$ = 1/T $_{\underline{s}}$ is the sampling frequency in samples per second. signals are formed by periodically
tinuous-time signal
 $x(n) = x_a(nT_s)$
acing T_s is the sampling period, and
sampling frequency in samples per
s-time signal is first multiplied by a
nce of impulses,
 $\sum_a(t) = \sum_{n=-\infty}^{\infty} \delta(t$ **Solution 3 and solution**
 Solution 3 and s the sampling frequency in samples per
 Solution 4 a signal is first multiplied by a
 Solution 4 a signal is first multiplied by a
 Solution 4 a signal
 Solution 4 a sig
- The continuous-time signal is first multiplied by a periodic sequence of impulses,

$$
S_a(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)
$$

• To form the sampled signal

$$
x_{s}(t) = x_{a}(t) s_{a}(t) = \sum_{n=-\infty}^{\infty} x_{a}(nT_{s}) \delta(t - nT_{s})
$$

Periodic Sampling

• The sampled signal is converted into a discretetime signal by mapping the impulses that are spaced in time by T_s into a sequence $x(n)$ where s converted into a discrete-
ing the impulses that are
into a sequence $x(n)$ where
ie indexed by the integer
 $y = x(nT)$ the sample values are indexed by the integer variable n:

$$
x(n) = x_a(nT_s)
$$

• This process is illustrated in Fig. (b).

• The Fourier transform of $\delta(t-nT_s)$ is $e^{-jn\Omega T_s}$, so the Fourier transform of the sampled signal $x_s(t)$ is

$$
X_{s}(j\Omega) = \sum_{n=-\infty}^{\infty} x_{a}(nT_{s})e^{-jn\Omega T_{s}}
$$
 (1)

amples	...
• The Fourier transform of $\delta(t)$ is:	
$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$	

• Because
$$
\int_{-\infty}^{\infty} \phi(t)\delta(t) dt = \phi(0)
$$

• The Fourier transform of $\delta(t-nT_s)$ is $e^{-jn\Omega T_s}$.

Periodic Sampling

• Another expression for $X_s(j\Omega)$ follows by noting that the Fourier transform of $s_a(t) = \sum^{\infty} \delta(t-nT_s)$ is n $s_a(t) = \sum_{r=0}^{\infty} \delta(t - nT_s)$ $=\sum_{n=-\infty}^{\infty}\delta(t-n)$ **pling**

Solid the same of $X_s(j\Omega)$ follows by noting
 2π transform of $s_a(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$ is
 $(3\Omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$
 T_s is the sampling frequency in

$$
S_a(j\Omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)
$$

- where $\Omega_s = 2\pi/T_s$ is the sampling frequency in radians per second.
- By multiplication property of Fourier transform, nother expression for $X_s(j\Omega)$ follows by noting

at the Fourier transform of $s_s(t) = \sum_{n=s}^{\infty} \delta(t-nT_s)$ is
 $S_a(j\Omega) = \frac{2\pi}{T_s} \sum_{k=s\infty}^{\infty} \delta(\Omega - k\Omega_s)$

nere $\Omega_s = 2\pi/T_s$ is the sampling frequency in

dians per second.

m \overline{c} jn ΩT_s $s_S(J\Omega) = \sum_{a} x_a (nI_s) e^{-s} = \frac{1}{2\pi} \Lambda_a (J\Omega)^* S_a (J\Omega) = \frac{1}{T} \sum_{a} \Lambda_a (J\Omega - J\Omega_s)$ $\overline{n} = -\infty$ 2π I_s $\overline{k} =$ $X_{s}(j\Omega) = \sum x_{a}(nT_{s})e^{-jn\Omega T_{s}} = \frac{1}{2}X_{a}(j\Omega)*S_{a}(j\Omega) = \frac{1}{2}\sum X_{a}(j\Omega-jk\Omega)$ π ^{--a}(J --) π _a(J --) T_s ∞ and ∞ $-jn\Omega T$ $\mathcal{L}(\Omega) = \sum_{n=-\infty}^{\infty} x_a (nT_s) e^{-jn\Omega T_s} = \frac{1}{2\pi} X_a (j\Omega) * S_a (j\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a (j\Omega - jk\Omega_s)$
- Note that the above equation is derived from the Fourier analysis of the continuous time signal see pages 55 to 59).

Periodic Sampling

Finally, the discrete-time Fourier transform of $x(n)$: ∞ ∞

Sampling
\nthe discrete-time Fourier transform of
$$
x(n)
$$
:
\n
$$
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\omega} = \sum_{n=-\infty}^{\infty} x_a(nT_s)e^{-jn\omega}
$$
\n(2)

 $X\!\left(e^{\mathit{j}\omega}\right)$ is periodic with period 2 $\boldsymbol{\pi}$ (see page 37)

Comparing Eq. (1) with Eq. (2) , it follows that: (if ω in Eq.(2), set to ΩT_s , Eqs (1) and (2) will be same) F

time Fourier transform of $x(n)$:
 $x_1e^{-jn\omega} = \sum_{n=-\infty}^{\infty} x_a(nT_s)e^{-jn\omega}$ (2)

with period 2 π (see page 37)

ith Eq. (2), it follows that:

, Eqs (1) and (2) will be same)
 $\lim_{n\to\infty} \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a \left(j \frac{\omega}{$ **ampling**

a discrete-time Fourier transform of $x(n)$:
 $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\omega} = \sum_{n=-\infty}^{\infty} x_a(nT_s)e^{-jn\omega}$ (2)
 $(e^{j\omega})$ is periodic with period 2π (see page 37)

ng Eq. (1) with Eq. (2), it follows that:

(2),

$$
X\left(e^{j\omega}\right) = X_s\left(j\Omega\right)\Big|_{\Omega = \omega/T_s} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a\left(j\frac{\omega}{T_s} - j\frac{2\pi k}{T_s}\right) \tag{3}
$$

• Thus, $X(e^{j\omega})$ is a frequency-scaled version of $X_s(j\Omega)$, with the scaling defined by

$$
\omega = \Omega T_s
$$

• This scaling, which makes $X(e^{j\omega})$ periodic with a period of 2π , is a consequence of the time-scaling that occurs when $x_s(t)$ is converted to $x(n)$. (In Eq. (3), $X_a\left(j\frac{\omega}{T}-j\frac{2\pi k}{T}\right)$ has 2 π period) $a\left(\sqrt{\frac{r}{T_s}}-J\right)\frac{r}{T_s}$ $X_a \left(j \frac{\omega}{\pi} - j \frac{2 \pi k}{\pi} \right)$ $\overline{T_s}$ $\overline{T_s}$ $\begin{pmatrix} a & 2\pi k \end{pmatrix}$ $j\frac{\omega}{T}-j\frac{2\pi\kappa}{T}$ | $\begin{pmatrix} 0 & T_s & T_s \end{pmatrix}$ $X(e^{j\omega}) = X_s(j\Omega)|_{\Omega = \omega_{iT_s}} = \frac{1}{T_s} \sum_{k=s} X_a \left(\frac{j}{T_s} - j \frac{2\pi k}{T_s} \right)$ (3)
 hus, $X(e^{j\omega})$ **is a frequency-scaled version of** $X_s(j\Omega)$,
 with the scaling defined by
 $\omega = \Omega T_s$

This scaling, which makes $X(e^{j\omega})$ peri

Note that the range of ω is [0 2π], $2\pi/T_s$ is sampling frequency

Sampling Theorem

• If $x_a(t)$ is strictly bandlimited,

$$
X_a(j\Omega) = 0 \qquad |\Omega| > \Omega_0
$$

• Then $x_a(t)$ may be uniquely recovered from its sampled signal $x_{\scriptscriptstyle S}(t)$ if

$$
\Omega_s = \frac{2\pi}{T_s} \ge 2\Omega_0
$$

• If $x_a(t)$ is strictly bandlimited,
 $x_a(t) = 0$ |Q|> Ω_0

• Then $x_a(t)$ may be uniquely recovered from its

sampled signal $x_s(t)$ if
 $\Omega_s = \frac{2\pi}{T_s} \ge 2\Omega_0$

• The frequency Ω_0 is called the Nyquist frequency,

a and the minimum sampling frequency, $\Omega_s = 2\Omega_0$, is If $x_a(t)$ is strictly bandlimited,
 $x_a(j\Omega) = 0$ | Ω | $> \Omega_0$

Then $x_a(t)$ may be uniquely recovered from its

sampled signal $x_s(t)$ if
 $\Omega_s = \frac{2\pi}{T_s} \ge 2\Omega_0$

The frequency Ω_0 is called the Nyquist frequency,

and The frequency Ω_0 is called the Nyquist frequency,

Find the Fourier transform of the periodic impulse train of $\delta(t-nT_s)$: **ES**
 EQUATE:
 EQUATE:
 EQUATE:
 EQUATE:
 $\delta(t-nT_s)$:
 $\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT_0)$

the inverse Fourier transform definition, we ∞ nsform of the periodic impulse
 $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$

Fourier transform definition, we
 $(\omega)e^{j\omega t}d\omega = 2\pi x(t)$

we obtain nsform of the periodic impulse
 $x) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$

Fourier transform definition, we
 $(\omega)e^{j\omega t}d\omega = 2\pi x(t)$

we obtain
 $(\omega)e^{-j\omega t}d\omega = 2\pi x(-t)$

ng t and ω , we get

$$
\delta_{T_0}\left(t\right)=\sum_{k=-\infty}^{\infty}\delta\left(t-kT_0\right)
$$

• From the inverse Fourier transform definition, we have t) = $\sum_{k=-\infty} \delta(t - kT_0)$

Fourier transform definition, we

(ω) $e^{j\omega t} d\omega = 2\pi x(t)$

we obtain

(ω) $e^{-j\omega t} d\omega = 2\pi x(-t)$

g t and ω , we get

(t) $e^{-j\omega t} dt = 2\pi x(-\omega)$
 $X(t)$) = $\int_0^\infty X(t) e^{-j\omega t} dt$

$$
\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = 2\pi x(t)
$$

- Changing t to $-t$, we obtain $\int_{0}^{\infty} X(\omega)e^{-j\omega t} d\omega = 2\pi x(-t)$ $\int_{-\infty}^{\infty} X(\omega) e^{-j\omega t} d\omega = 2\pi x(-t)$
- Now interchanging t and ω , we get $\int_{0}^{\infty} X(t) e^{-j\omega t} dt = 2\pi x(-\omega)$ $\int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt = 2\pi x(-\omega)$ $\chi(\omega)e^{j\omega t}d\omega = 2\pi x(t)$

we obtain
 $\chi(\omega)e^{-j\omega t}d\omega = 2\pi x(-t)$

ng t and ω , we get
 $\chi(t)e^{-j\omega t}dt = 2\pi x(-\omega)$
 $\{\chi(t)\} = \int_{-\infty}^{\infty} X(t)e^{-j\omega t}dt$
- Since

$$
\mathscr{F}\left\{X(t)\right\}=\int_{-\infty}^{\infty}X(t)e^{-j\omega t}dt
$$

- We get the following duality property: **amples**
• We get the following duality property:
 $X(t) \leftrightarrow 2\pi x(-\omega)$
• We know that the Fourier transform of $\partial(t)$ is 1.
• Thus, by the duality property we get
 $1 \leftrightarrow 2\pi \delta(-\omega) = 2\pi \delta(\omega)$ $X(t) \leftrightarrow 2\pi x(-\omega)$ duality property:
 $2\pi x(-\omega)$

urier transform of $\delta(t)$ is 1.

roperty we get
 $-\omega$) = $2\pi\delta(\omega)$
 $(t) = \mathcal{F}^{-1}{X(\omega)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$

lowing equation:
-
- Thus, by the duality property we get

 $1 \leftrightarrow 2\pi\delta(-\omega) = 2\pi\delta(\omega)$

- And, we know that $x(t) = \mathcal{F}^{-1}\left\{t\right\}$ 2 $g(x(t) = \mathscr{F}^{-1}\left\{X(\omega)\right\} = \frac{1}{2\pi} \int_{0}^{\infty} X(\omega)e^{j\omega t} d\omega$ π $-1(\mathbf{v}(\omega)) = \frac{1}{\omega} \mathbf{r}^{\omega}$ $=\mathscr{F}^{-1}\left\{X\left(\omega\right)\right\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}$
- We can derive the following equation:

$$
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \delta(\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t} \int_{-\infty}^{\infty} \delta(0) d\omega = e^{j\omega_0 t}
$$

Then we get

$$
e^{j\omega_0 t} \leftrightarrow 2\pi \delta(\omega - \omega_0)
$$
 (4)

- We express periodic signal $x(t)$ as periodic signal $x(t)$ as
 $(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ $\omega_0 = \frac{2\pi}{T_0}$

ourier transform of both sides and

and the linearity property we get $\overline{0}$ 0 jk $\omega_0 t$ 2 k k $x(t) = \sum_{k} c_{k} e^{jka}$ T_{0} $\omega_0 t$ 2π $\omega_{\scriptscriptstyle{\alpha}}$ ∞ $=-\infty$ $= \sum_{k=1}^{\infty} c_k e^{jk\omega_0 t}$ $\omega_0 = \frac{2}{\sqrt{3}}$
- Taking the Fourier transform of both sides and using Eq. (2) and the linearity property, we get Findic signal $x(t)$ as
 $=\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_k t}$ $\omega_0 = \frac{2\pi}{T_0}$

Therefore transform of both sides and
 $\omega_0 = 2\pi \sum_{k=-\infty}^{\infty} c_k \delta(\omega - k\omega_0)$
 $\omega_0 = 2\pi \sum_{k=-\infty}^{\infty} c_k \delta(\omega - k\omega_0)$
 $\omega_0 = 2\pi \sum_{k=-\infty}^{\infty} c_k \delta(\omega$ periodic signal $x(t)$ as
 $(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ $\omega_0 = \frac{2\pi}{T_0}$

ourier transform of both sides and

and the linearity property, we get
 $X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} c_k \delta(\omega - k\omega_0)$
 x exponential Fourier series o

$$
X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} c_k \delta(\omega - k\omega_0)
$$

• The complex exponential Fourier series of $\delta_{T_0}(t)$ is given by:

$$
\delta_{T_0}\left(t\right) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \qquad \omega_0 = \frac{2\pi}{T_0}
$$

• The Fourier coefficients can be obtained since $\delta(t)$ is involved: transform of both sides and

le linearity property, we get
 $2\pi \sum_{k=-\infty}^{\infty} c_k \delta(\omega - k\omega_0)$

lential Fourier series of $\delta_{r_0}(t)$ is
 $c_k e^{jk\omega_0 t}$ $\omega_0 = \frac{2\pi}{T_0}$

lents can be obtained since $\delta(t)$
 $\delta^{(2)}$
 $\delta(t)$

$$
c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0}
$$

$\delta(\omega - k\omega_0)$
 $(\omega - k\omega_0)$

of a unit impulse train

Examples

• Finally, we get

get
\n
$$
\mathscr{F}\left[\delta_{T_0}(t)\right] = \frac{2\pi}{T_0} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\omega_0\right)
$$
\n
$$
= \omega_0 \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\omega_0\right)
$$

• Thus, the Fourier transform of a unit impulse train is also a similar impulse train.

Examples: Convolution Calculation

• By convolution definition, we get

$$
x(t) * \delta(t - t_0) = \delta(t - t_0) * x(t)
$$

=
$$
\int_{-\infty}^{\infty} \delta(\tau - t_0) x(t - \tau) d\tau = x(t - \tau)|_{\tau = t_0}
$$

=
$$
x(t - t_0)
$$

- We have seen how to represent a sequence in terms of a linear combination of complex exponentials using the discrete-time Fourier transform (DTFT).
- For finite-length sequences there is another representation, called the discrete Fourier transform (DFT).
- Unlike the DTFT, which is a continuous function of a continuous variable, ω , the DFT is a sequence that corresponds to samples of the DTFT.
- Such a representation is very useful for digital computations.

- The DFT may be easily developed from the discrete Fourier series for periodic sequences.
- Let $x(n)$ be a finite-length sequence of length N that is equal to zero outside the interval [0, N-1].
- A periodic sequence $\tilde{x}(n)$ may be formed from $x(n)$ as follows: **EXECUTE THE INCREDIBLE 11 ATTSTUTTIFF 11 ATTSTUTTIFF 11 ATTSTUTTIFF 11 ATTSTUTTIFF 11 ATTSTURE 11 ATTSTURE 11 ATTSTURE 11 ATTSTURE 2013 (1) ATTSTURE 2013 (1) ATTSTURE 2013 (1) ATTSTURE 2013 (1) ATTSTURE 11 ATTSTURE 11 AT**

$$
\tilde{x}(n) = \sum_{k=-\infty}^{\infty} x(n + kN)
$$

• A periodic sequence may be expanded using the DFS as Eq. (1).

Figure Series for periodic sequences.
finite-length sequence of length *N*
to zero outside the interval [0, *N*-1].
quence
$$
\tilde{x}(n)
$$
 may be formed from $x(n)$

$$
\tilde{x}(n) = \sum_{k=-\infty}^{\infty} x(n+kN)
$$
equence may be expanded using the
).

$$
\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) e^{j2\pi nk/N}
$$
 (1)

• Fourier series coefficients, $\tilde{X}(k)$ may be derived by multiplying both sides of this expansion by $e^{-j2\pi nl/N}$ summing over one period and using the fact that the complex exponentials are orthogonal: efficients, $\tilde{X}(k)$ may be derived by

h sides of this expansion by $e^{-j2\pi n l/N}$
 ne period and using the fact that
 onentials are orthogonal:
 $e^{j2\pi n(k-l)/N} = \begin{cases} N & k = l \\ 0 & k \neq l \end{cases}$
 $(k) = \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j2\pi$

$$
\sum_{k=0}^{N-1} e^{j2\pi n(k-l)/N} = \begin{cases} N & k=l\\ 0 & k \neq l \end{cases}
$$

• The result is

$$
\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j2\pi nk/N}
$$

• Because $x(n) = \tilde{x}(n)$ for $n = 0, 1, ..., N-1, x(n)$ may similarly be expanded as follows:

Fourier Transform

\n**e**
$$
x(n) = \tilde{x}(n)
$$
 for $n = 0, 1, \ldots, N-1$, $x(n)$ may

\n y be expanded as follows:

\n
$$
x(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) e^{j2\pi nk/N} \quad 0 \leq n < N
$$

\n
$$
\tilde{X}(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \quad 0 \leq k < N
$$

\n**e the DFS coefficients are periodic, if we**

\n**be one period of** $\tilde{X}(k)$ and replace $\tilde{X}(k)$ in

\n**with** $X(k)$, then we have

\n
$$
x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi nk/N} \quad 0 \leq n < N
$$

\n**Supence** $X(k)$ is called the N-point DFT of $x(n)$.

• Because the DFS coefficients are periodic, if we Because $x(n) = \tilde{x}(n)$ for $n = 0, 1, ..., N$ -1, $x(n)$ may

similarly be expanded as follows:
 $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) e^{j2\pi nk/N}$ $0 \le n < N$
 $\tilde{X}(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$ $0 \le k < N$
 Because the DFS coefficients ar the sum with $X(k)$, then we have • $N_{k=0}^{N-1}$
 $\tilde{X}(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}$ $0 \le k < N$

• **Because the DFS coefficients are periodic, if we let** $X(k)$ **be one period of** $\tilde{X}(k)$ and replace $\tilde{X}(k)$ in the sum with $X(k)$, then we have $x(n) = \frac{1$

$$
x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi nk/N} \qquad 0 \le n < N
$$

screte Fourier Transform
• These coefficients are related to $x(n)$ as follows:
 $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}$ 0 ≤ k < N Discrete Fourier Transform

Ourier Transform

\nefficients are related to
$$
x(n)
$$
 as follows:

\n $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$ $0 \le k < N$

\nlog the definition of the DFT of $x(n)$ to the

**SCrete Fourier Transform

•** These coefficients are related to $x(n)$ as follows:
 $x(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}$ $0 \le k < N$

• Comparing the definition of the DFT of $x(n)$ to the DFT, it follows that the DFT coefficients ar DTFT, it follows that the DFT coefficients are These coefficients are related to $x(n)$ as foll
 $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$ $0 \le k < N$
 Comparing the definition of the DFT of x(r
 DTFT, it follows that the DFT coefficients a
 samples of the DTFT:
 $X(k) = \sum_{n=0}$ coefficients are related to $x(n)$ as follows:
 $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}$ $0 \le k < N$

aring the definition of the DFT of $x(n)$ to the

it follows that the DFT coefficients are
 $\frac{\log x}{\log x}$ (n) $e^{-j2\pi nk/N} = \sum_{n=-\infty}^{\infty}$

$$
X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi nk/N} = X(e^{j\omega})|_{\omega=2\pi k/N}
$$

Note that DFT has a periodic property so that we extract the values in the range of [0, N-1].

DFT Circular Shift Property

- The DFT has a circular shift property.
- **Circular Shift Property**

 The DFT has a circular shift property.

 Suppose that the values of a sequence $x(n)$, from
 $n=0$ to $n=N-1$, are marked around a circle as

illustrated in Figure (a). $n=0$ to $n=N-1$, are marked around a circle as illustrated in Figure (a).

(a) An eight-point sequence.

 (b) Circular shift by two.

• A circular shift to the right by n_{o} corresponds to a rotation of the circle n_o positions in a clockwise direction as shown in Figure (b).

DFT Circular Shift Property

- Another way to circularly shift a sequence is to form the periodic sequence $\tilde{x}(n)$, perform a linear shift, $\tilde{x}(n-n_0)$ and then extract one period of $\tilde{x}(n-n_0)$ by multiplying by a rectangular window. • **Circular Shift Property**
• Another way to circularly shift a sequence is to
torm the periodic sequence $\tilde{x}(n)$, perform a linear
shift, $\tilde{x}(n-n_0)$ and then extract one period of $\tilde{x}(n-n_0)$
by multiplying by a rec
- follows:

 $x((n-n₀))_n R_N(n) = \tilde{x}(n-n₀) R_N(n)$

• where $n_{\rm o}$ is the amount of the shift and $\mathcal{R}_{N}(n)$ is a rectangular window.

DFT Circular Shift Property

• Examples illustrating the circular shift of a fourpoint sequence are shown in Figures (a), (b). (c), and (d).

(a) A discrete-time signal of length $N = 4$.

 (b) Circular shift by one.

DFT Circular Convolution

- **Let** $h(n)$ **and** $x(n)$ **be finite-length sequences of

length N with N-point DFTs** $H(k)$ **and** $X(k)$ **,

respectively.** length N with N-point DFTs $H(k)$ and $X(k)$, respectively. (*n*) and $x(n)$ be finite-length sequences of

h *N* with *N*-point DFTs *H*(*k*) and *X*(*k*),

cctively.

equence that has a DFT equal to the product
 $H(k)X(k)$ is
 $(n) = \left[\sum_{k=0}^{N-1} \tilde{h}(k)\tilde{x}(n-k)\right] R_{N}(n) = \left[\sum_{k=0}^{N-1$
- The sequence that has a DFT equal to the product $Y(k)=H(k)X(k)$ is

$$
y(n) = \left[\sum_{k=0}^{N-1} \tilde{h}(k)\tilde{x}(n-k)\right] R_N(n) = \left[\sum_{k=0}^{N-1} \tilde{h}(n-k)\tilde{x}(k)\right] R_N(n) \tag{2}
$$

• where $\tilde{x}(n)$ and $\tilde{h}(n)$ are the periodic extensions of the sequences $x(n)$ and $h(n)$, respectively.

DFT Circular Convolution

Because $\tilde{h}(n) = h(n)$ for $0 \le n \le N$, the sum in Eq. (2) may also be written as ar Convolution
 $\tilde{h}(n) = h(n)$ for $0 \le n < N$, the sum in Eq. (2)

be written as
 $y(n) = \left[\sum_{k=0}^{N-1} h(k) \tilde{x}(n-k)\right] R_{N}(n)$ (3)

ence $y(n)$ in Eq. (3) is the N-point circular

$$
y(n) = \left[\sum_{k=0}^{N-1} h(k)\tilde{x}(n-k)\right] \mathcal{R}_N(n)
$$
 (3)

• The sequence y(n) in Eq. (3) is the N-point circular convolution of $h(n)$ with $x(n)$, and it is written as $y(n) = \left[\sum_{k=0} h(k) \tilde{x}(n-k) \right] R_{N}(n)$

ience $y(n)$ in Eq. (3) is the *N*-point circ

ion of $h(n)$ with $x(n)$, and it is written
 $y(n) = h(n)$ \bigcirc $x(n) = x(n)$ \bigcirc $h(n)$

Circular Versus Linear Convolution

- In general, circular convolution is not the same as linear convolution.
- However, there is a relationship between circular and linear convolution that illustrates what steps must be added to ensure that they are the same. • In general, circular convolution is not the same as

linear convolution.

• However, there is a relationship between circular

and linear convolution that illustrates what steps

must be added to ensure that they are th **linear convolution.**

However, there is a relationship between circular

and linear convolution that illustrates what steps

must be added to ensure that they are the same

Specifically, let $x(n)$ and $h(n)$ be finite-len here is a relationship between circular
convolution that illustrates what steps
ded to ensure that they are the same.
let $x(n)$ and $h(n)$ be finite-length
and let $y(n)$ be the linear convolution:
 $y(n)=x(n)*h(n)$
t circular co
- Specifically, let $x(n)$ and $h(n)$ be finite-length sequences and let $y(n)$ be the linear convolution:

 $y(n) = x(n) * h(n)$

$$
h(n) \bigcirc \mathbb{D} x(n) = \left[\sum_{k=-\infty}^{\infty} y(n + kN) \right] \mathcal{R}_N(n)
$$
 (4)

The circular convolution of two sequences is found by performing the linear convolution and Specifically, let $x(n)$ and $h(n)$ be finite-leng
sequences and let $y(n)$ be the linear convo
 $y(n) = x(n) * h(n)$
The N-point circular convolution of $x(n)$ wi
related to $y(n)$ as follows:
 $h(n) \bigotimes x(n) = \left[\sum_{k=-\infty}^{\infty} y(n+kN)\right] \mathcal{R$

Circular Versus Linear Convolution

Example: Let us find the four-point circular convolution of the sequences $h(n)$ and $x(n)$: the linear convolution is $y(n) = \delta(n) + \delta(n-1) + 2\delta(n-2) + 2\delta(n-3) + 3\delta(n-5)$ **EXECUTE SIMURE:**
 EXECUTE SIMURE:

The sequences $h(n)$ and $x(n)$: the linear

is $y(n)=\delta(n)+\delta(n-1)+2\delta(n-2)+2\delta(n-3)+3\delta(n-5)$

et up a table to evaluate the sum

(*n*) $\bigcirc x(n)=\left[\sum_{k=0}^{\infty}y(n+kN)\right]R_k(n)$

me by listing the value

• We may set up a table to evaluate the sum

$$
h(n) \bigotimes x(n) = \left[\sum_{k=-\infty}^{\infty} y(n + kN) \right] R_{N}(n)
$$

- This is done by listing the values of the sequence $y(n + kN)$ in a table and summing these values for $n = 0,1,2,3.$
- Thus, we have $n \mid 0, 1, 2, 3 \mid 4, 5, 6, 7$ $1 \quad 1 \quad 2 \quad 2 \quad 0 \quad 3 \quad 0 \quad 0$ $0 \t3 \t0 \t0 \t0 \t0 \t0 \t0$ Le roar point encarration

uences $h(n)$ and $x(n)$: the linear
 $h(x) + \delta(n-1) + 2\delta(n-2) + 2\delta(n-3) + 3\delta(n-5)$

ble to evaluate the sum
 $\sum_{k=-\infty}^{\infty} y(n+kN) \bigg] R_{y}(n)$

ang the values of the sequence

and summing these values fo (n) + δ (n-1) + 2δ (n-2) + 2δ (n-3) + 3δ (n-5)

able to evaluate the sum
 $\left[\sum_{k=-\infty}^{\infty} y(n+kN)\right] R_{N}(n)$

ng the values of the sequence

and summing these values for
 $\frac{n}{y(n)}$
 $\frac{0}{1}$ $\frac{1}{1}$ $\frac{2}{2}$ $\frac{$ be to evaluate the sum
 $=\left[\sum_{k=-\infty}^{\infty} y(n+kN)\right] R_{\nu}(n)$

ting the values of the sequence

e and summing these values for
 $\frac{n}{y(n)}$
 $\frac{0}{1}$
 $\frac{1}{1}$
 $\frac{2}{2}$
 $\frac{3}{2}$
 $\frac{4}{0}$
 $\frac{5}{3}$
 $\frac{6}{0}$
 $\frac{7}{y(n+4$ $\overline{4}$ n $y(n)$ $y(n+$ $h(n) \overline{a(x(n))}$ $+$ \bigoplus
- Summing the columns for $0 \le n \le 3$, we have

 $h(n) \oplus x(n) = \delta(n) + 4\delta(n-1) + 2\delta(n-2) + 2\delta(n-3)$

Circular Versus Linear Convolution

• An important property that follows from Eq. (4) is that if $y(n)$ is of length N or less, circular convolution is equivalent to linear convolution.

 $h(n) \bigotimes x(n) = h(n) * x(n)$

Thus, if $h(n)$ **and** $x(n)$ **are finite-length sequences** of length N_1 and N_2 , respectively, $y(n) = h(n) * x(n)$ Fire and Correction

trively that follows from Eq. (4) is

the N or less, circular

valent to linear convolution.
 $x(n) = h(n) * x(n)$

(a) are finite-length sequences

the N-point circular

valent to linear convolution

valent of length $N_1 + N_2 - 1$, and the N-point circular convolution is equivalent to linear convolution provided $N \ge N_1 + N_2 - 1$.

Linear Convolution Using DFT

- The DFT provides a convenient way to perform convolutions without the convolution sum.
- **Fraction Specifically, if** $h(n)$ **is N₂ points long and x(n) is N₂
** \bullet **The DFT provides a convenient way to perform

convolutions without the convolution sum.
** \bullet **Specifically, if** $h(n)$ **is N₁ points long and x(n) ar Convolution Using DFT**

The DFT provides a convenient way to perform

convolutions without the convolution sum.

Specifically, if $h(n)$ is N_1 points long and $x(n)$ is N_2

points long, $h(n)$ may be linearly convo **ar Convolution Using DFT**
The DFT provides a convenient way to
convolutions without the convolution
Specifically, if $h(n)$ is N_1 points long and
points long, $h(n)$ may be linearly convo
x(n) as follows:
1. <u>Pad the se</u> **ar Convolution Using DFT**

The DFT provides a convenient way to perform

convolutions without the convolution sum.

Specifically, if $h(n)$ is N_1 points long and $x(n)$ is N_2

points long, $h(n)$ may be linearly convo The DFT provides a convenient way to perform
convolutions without the convolution sum.
Specifically, if $h(n)$ is N_1 points long and $x(n)$ is N_2
points long, $h(n)$ may be linearly convolved with
 $x(n)$ as follows:
1. Specifically, if $h(n)$ is N_1 points long anc
points long, $h(n)$ may be linearly convo
 $x(n)$ as follows:
1. <u>Pad the sequences $h(n)$ **and** $x(n)$ **with the inverse of length** $N \ge N_1 + N_2 - 1$.
2. Find the N-point DFTs of </u>
	- that they are of length $N \ge N_1 + N_2 1$.
	-
	-
	-
- Significant computational savings for DFT may be realized with the fast Fourier transform (FFT).

2D Fourier Transform Definitions

• The analytic Fourier transform of a function g of two variables x and y is given by: **rier Transform Definitions**

analytic Fourier transform of a function g of

variables x and y is given by:
 $G(f_x, f_y) = \int_a^{\infty} [g(x, y) \exp[-j2\pi(f_x x + f_y y)] dx dy$,

re $G(f_x, f_y)$ is the transform result and f_x and f_y

independent ∞

$$
G(f_X, f_Y) = \int_{-\infty}^{\infty} \int g(x, y) \exp[-j2\pi (f_X x + f_Y y)] dx dy,
$$

- where $G(f_x, f_y)$ is the transform result and f_x and f_y are independent frequency variables associated with x and y , respectively. intriables x and y is given by:
 f_x, f_y = $\int_{\infty}^{\infty} \int g(x, y) \exp[-j2\pi (f_x x + f_y y)] dx dy$,
 $G(f_x, f_y)$ is the transform result and f_x and f_y

dependent frequency variables associated

and y, respectively.

peration is often d
- This operation is often described in a shorthand manner as $\mathfrak{I}\{g(x,y)\}=G(f_x,f_y).$
- The analytic inverse Fourier transform is given by: $= \int_{a}^{\infty} \int G(f_X, f_Y) \exp[j2\pi (f_X x + f_Y y)] df$ $-\infty$
- The shorthand notation for this operation is

 $\mathfrak{I}^{-1}{G(f_X, f_Y)} = g(x, y).$

Discrete Fourier Transform from Continuous Transform **Example 3 . Follow Transform from**
 Solution Stransform
 Solution Stransform of a function g of

variables x and y is repeated for reference:
 $G(f_x, f_y) = \int_{-\infty}^{\infty} [g(x, y) \exp[-j2\pi (f_x x + f_y y)] dx dy.$ (5)

, assume $g(x, y)$ is

The analytic Fourier transform of a function g of two variables x and y is repeated for reference:

$$
G(f_X, f_Y) = \int_{-\infty}^{\infty} \int g(x, y) \exp[-j2\pi (f_X x + f_Y y)] dxdy.
$$
 (5)

- First, assume $g(x,y)$ is sampled as $g(m\Delta x, n\Delta y)$ $g(m\Delta x, n\Delta y) \rightarrow \tilde{g}(m, n)$.
- The integrals in Eq. (5) can be approximated using a Riemann sum:

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots dx dy \rightarrow \sum_{n=-N/2}^{N/2-1} \sum_{m=-M/2}^{M/2-1} \dots \Delta x \Delta y.
$$

The convention for the frequency domain is to <u>divide this continuous space indicated by f_{χ} and f_{γ} .</u> \mathbf{r} into M and N evenly spaced coordinate values as follows (M, N: total number of samples)

$$
f_X \rightarrow \frac{p}{M\Delta x}, \quad where \quad p = -\frac{M}{2}, ..., \frac{M}{2} - 1;
$$

\n
$$
f_Y \rightarrow \frac{q}{N\Delta y}, \quad where \quad q = -\frac{N}{2}, ..., \frac{N}{2} - 1;
$$
\n(6)

• where p and q are integers that index multiples of the frequency sample intervals.

$$
\Delta f_X = \frac{1}{M \Delta x} = \frac{1}{L_X}, \quad and \quad \Delta f_Y = \frac{1}{N \Delta y} = \frac{1}{L_Y}.
$$
 (6-1)

• In fact, p and q take on the same values as m and n , respectively, since the spatial and frequency arrays have the same number of elements.

- Note that the maximum absolute values of the frequency coordinates in Eq. (6) are the Nyquist <u>frequencies $1/(2\Delta x) = f_{NX}$ and $1/(2\Delta y) = f_{NY}$.</u>
- Incorporating Eq. (6) into the complex exponential kernel of Eq. (5) yields

exp 2 exp 2 exp 2 . X Y p q j f x f y j m x n y M x N y pm qn ^j M N

Discrete Fourier Transform from Continuous Transform **Fourier Transform from**
 Solution Stransform

we arrive at the following form of the DFT:
 $(p,q) = \sum_{m=M/2}^{M/2-1} \sum_{n=N/2}^{N/2-1} \tilde{g}(m,n) \exp \left[-j2\pi \left(\frac{pm}{M} + \frac{qn}{N}\right)\right],$ (7)

verse discrete Fourier transform (DFT⁻¹) is

• Finally, we arrive at the following form of the DFT:

IONS Transform
\nly, we arrive at the following form of the DFT:
\n
$$
\tilde{G}(p,q) = \sum_{m=-M/2}^{M/2-1} \sum_{n=-N/2}^{N/2-1} \tilde{g}(m,n) \exp\left[-j2\pi \left(\frac{pm}{M} + \frac{qn}{N}\right)\right],
$$
\n
$$
\text{inverse discrete Fourier transform (DFT-1) is}
$$
\n
$$
\tilde{g}(m,n) = \frac{1}{MN} \sum_{p=-N/2}^{M/2-1} \sum_{q=-M/2}^{N/2-1} \tilde{G}(p,q) \exp\left[j2\pi \left(\frac{pm}{M} + \frac{qn}{N}\right)\right].
$$
\n
$$
\text{forward and inverse DFTs are not usually}
$$
\n(8)

• The inverse discrete Fourier transform (DFT−1) is derived in a similar way and is written as:

$$
\tilde{g}(m,n) = \frac{1}{MN} \sum_{p=-N/2}^{M/2-1} \sum_{q=-M/2}^{N/2-1} \tilde{G}(p,q) \exp\bigg[j2\pi\bigg(\frac{pm}{M} + \frac{qn}{N}\bigg)\bigg].
$$
 (8)

The forward and inverse DFTs are not usually accomplished with a direct use of Eqs. (7), (8): they are accomplished with the computationally efficient FFT and FFT⁻¹ algorithms.

• Analytic rectangle function is shown in Fig. (a) (solid line) along with a sampled version (dots).

The periodic form of the function, which extends (virtually) beyond the original span of the sample vector, is also indicated (dashed line).

• Figure (b) shows the magnitude of the analytic spectrum of the rectangle (solid), the FFT result (dots), and the periodic spectrum (dashed).

- The most difference between the analytic and sample spectra in this case is slightly larger sample values in the magnitude at higher frequencies.
- This effect results from aliasing of under-sampled frequencies in the rectangle spectrum.

If the sampling frequency is $2\Omega_B$ where Ω_B is max frequency of the signal,

