Optical Signal Processing

Prof. Inkyu Moon

Dept. of Robotics Engineering, DGIST

Introduction

What is purpose?

- This course provides students with a basic understanding of the scientific principles associated with 1) Fourier optics, 2) image capture and formation, and 3) intelligent holographic imaging for numerous biomedical applications.
- **Digital holographic microscopy (DHM)** is also introduced in this course for three-dimensional (3D) and quantitative sensing, imaging and measuring of biological and microscopic samples.

What are the prerequisites?

- Fourier analysis & Basic optics

Introduction

Why are you in the class?

- Foundation for most 3D imaging system & modeling of a digital holographic microscope
- For MS or Ph.D. examinations.

Grading

- Midterm Exam: 40%
- Final Exams: 50%
- Homework/Class Participation: 10%

Reference

- J. Goodman, Introduction to Fourier Optics, Mcgraw-Hill, USA 1996

Chapter 1
Fourier Theory Review

A Little History and Purpose

- The study of **Optical Signal Processing** today leads naturally toward the computer for the following reason: **Fast Fourier transform (FFT) algorithm** provides an extremely efficient computational approach for solving wave optics problems.
- The FFT's speed makes it possible to perform thousands of optical propagation or imaging simulations in a reasonable amount of time.
- **The methods explored in this course form the basis for wave (or physical) optics simulation tools that are widely used in industry.**

A Little History and Purpose

- This course also provides step-by-step instructions for coding Fourier optics with MATLAB software.
- **<u>The end of this course, you can program basic Fourier optics problems—at least that's the goal!</u>**
- I encourage you to consult some references for basic Fourier theory.
- Recommended reading: Schaums Outline of Digital Signal Processing

Linear Systems and Nonlinear Systems

Unit Impulse Function

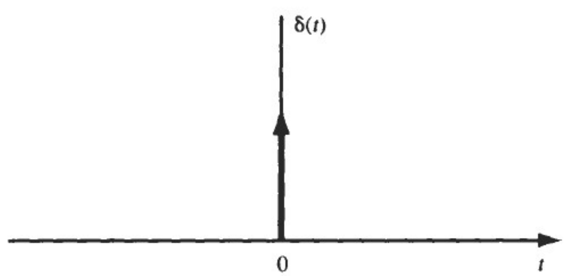
- The unit impulse function $\delta(t)$ plays a central role in system analysis: It has the following properties:

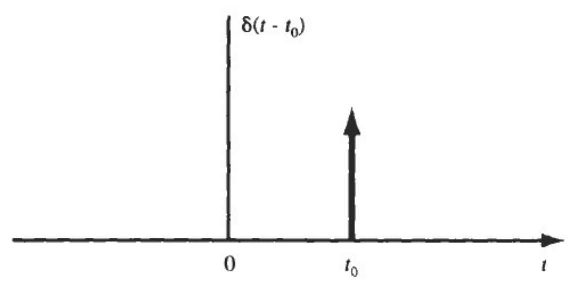
$$\delta(t)=\begin{cases}0 & t\neq 0\\ \infty & t=0\end{cases} \qquad \int_{-\varepsilon}^{\varepsilon}\delta(t)\,dt=1$$

- Thus, $\delta(t)$ cannot be an ordinary function and mathematically it is defined by

$$\int_{-\infty}^{\infty}\phi(t)\delta(t)\,dt=\phi(0)$$

- where $\phi(t)$ is any function continuous at $t = 0$.
- Similarly, the delayed delta function $\delta(t-t_0)$ is defined by $\int_{-\infty}^{\infty}\phi(t)\delta(t-t_0)\,dt=\phi(t_0)$





Linear Systems and Nonlinear Systems

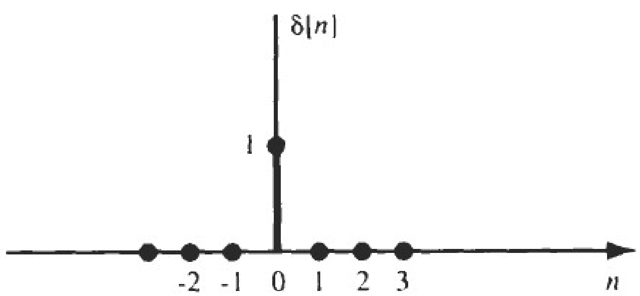
Unit Impulse Sequence (*n* is an integer)

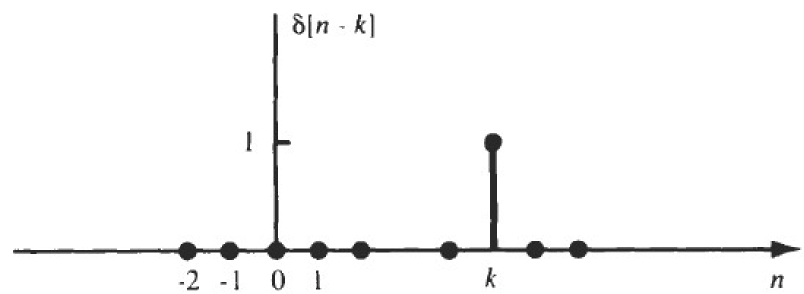
- The unit impulse (or unit sample) sequence $\delta[n]$ is defined:

$$\delta[n]=\begin{cases}1 & n=0\\ 0 & n\neq 0\end{cases}$$

- Similarly, the shifted unit impulse (or sample) sequence $\delta[n\text{-}k]$ is defined as

$$\delta[n-k]=\begin{cases}1 & n=k\\ 0 & n\neq k\end{cases}$$





Linear Systems and Nonlinear Systems

- If the operator *T* satisfies the following two conditions, then *T* is called a linear operator and the system represented by a linear operator *T* is called a linear system:

1. Additivity

- Given that $T\{x_1\} = y_1$ and $T\{x_2\} = y_2$ then $T\{x_1 + x_2\} = y_1 + y_2$ for any signals x_1 and x_2.

2. Homogeneity

- $T\{\alpha x\} = \alpha y$ for any signals *x* and any scalar α.
- Two equations $T\{x_1 + x_2\} = y_1 + y_2$ and $T\{\alpha x\} = \alpha y$ can be combined into a single condition as

$$\mathbf{T}\{\alpha_1 x_1 + \alpha_2 x_2\} = \alpha_1 y_1 + \alpha_2 y_2$$

Time-Invariant and Time-Varying Systems

- **A system is called time-invariant if a time shift (delay or advance) in the input signal causes the same time shift in the output signal**.
- For a continuous-time system, the system is time-invariant if $\mathbf{T}\{x(t-\tau)\} = y(t-\tau)$ for any real value of τ.
- For a discrete-time system, the system is time-invariant or shift-invariant if $\mathbf{T}\{x[n-k]\} = y[n-k]$ for any integer *k*.
- **A system which does not satisfy the above Equations is called a time-varying system**.

Linear Time-Invariant Systems

- **If the system is linear and time-invariant, then it is called a linear time-invariant (LTI) system.**
- **The input-output relationship for LTI systems is described in terms of a convolution operation.**
- The importance of the convolution operation in LTI systems stems from the fact that knowledge of the response of an LTI system to the unit impulse input allows us to find its output to any input signals.

Response of Continuous-Time LTI System

A. Impulse Response

- The impulse response $h(t)$ of a continuous-time LTI system (represented by T) is defined to be the response of the system when the input is $\delta(t)$:

$$h(t) = \mathbf{T}\{\delta(t)\}$$

B. Response to an Arbitrary Input

- **The input $x(t)$ can be expressed as**

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

- Since the system is linear, the response $y(t)$ of the system to the input $x(t)$ can be expressed as

$$\begin{aligned} y(t) &= \mathbf{T}\{x(t)\} = \mathbf{T}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau\right\} \\ &= \int_{-\infty}^{\infty} x(\tau)\mathbf{T}\{\delta(t-\tau)\}\,d\tau \end{aligned} \qquad (1)$$

Response of Continuous-Time LTI System

- Since the system is time-invariant, we have

$$h(t-\tau) = \mathbf{T}\{\delta(t-\tau)\} \tag{2}$$

- Substituting Eq. (2) into Eq. (1), we obtain

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau \tag{3}$$

- **Equation (3) indicates that a continuous-time LTI system is completely characterized by its impulse response *h*(*t*).**

Response of Continuous-Time LTI System

- Equation (3) defines the convolution of two continuous-time signals $x(t)$ and $h(t)$ denoted by

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \qquad (4)$$

- Equation (4) is called the *convolution integral*.
- **The output of any continuous-time LTI system is the convolution of the input $x(t)$ with the impulse response $h(t)$ of the system.**

Response of a Discrete-Time LTI System

A. Impulse Response

- The impulse response (or unit sample response) $h[n]$ of a discrete-time LTI system (represented by T) is defined to be the response of the system when the input is $\delta[n]$, that is,

$$h[n] = \mathbf{T}\{\delta[n]\}$$

B. Response to an Arbitrary Input

- **The input $x[n]$ can be expressed as**

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \qquad (5)$$

- Since the system is linear, the response $y[n]$ of the system to an input $x[n]$ can be expressed as:

$$y[n] = \mathbf{T}\{x[n]\} = \mathbf{T}\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}$$
$$= \sum_{k=-\infty}^{\infty} x[k]\mathbf{T}\{\delta[n-k]\} \qquad (6)$$

Response of a Discrete-Time LTI System

- Since the system is time-invariant, we have

$$h[n-k] = \mathbf{T}\{\delta[n-k]\} \qquad (7)$$

- Substituting Eq. (7) into Eq. (6), we obtain

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad (8)$$

- **Equation (8) indicates that a discrete-time LTI system is completely characterized by its impulse response *h*[*n*].**

Convolution Sum

- Equation (8) defines the convolution of two sequences $x[n]$ and $h[n]$ denoted by

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \qquad (9)$$

- Equation (9) is called the convolution sum.
- **The output of any discrete-time LTI system is the convolution of the input $x[n]$ with the impulse response $h[n]$ of the system.**

Frequency Response of Continuous-Time LTI Systems

- We know that the output $y(t)$ of a continuous-time LTI system equals the convolution of the input $x(t)$ with the impulse response $h(t)$: that is,

$$y(t) = x(t) * h(t)$$

- Applying the **convolution property**, we obtain

$$Y(\omega) = X(\omega) H(\omega)$$

- where $Y(\omega)$, $X(\omega)$, and $H(\omega)$ are the Fourier transforms of $y(t)$, $x(t)$, and $h(t)$, respectively.
- ***H*(*ω*) is called the system's frequency response**.

Fourier Analysis of Signals

- The Fourier representation of signals plays an extremely important role in both continuous-time and discrete-time signal processing.
- **We review the continuous-time Fourier transform (FT) and the FT for discrete-time signals).**
- Review Fourier series and Fourier transform which convert time-domain signals into frequency-domain (or spectral) representations.

Complex Exponential Signals

- The complex exponential signal $x(t) = e^{jw_0 t}$ is an important example of a complex signal.
- The fundamental period T_0 of $x(t)$ is given by:

$$T_0 = \frac{2\pi}{\omega_0}$$

- **Any signals can be expressed by using the complex exponential form.**
- The complex exponential sequence is of the form.

$$x[n] = e^{j\Omega_0 n}$$

- **Any sequences can be expressed by using the complex exponential form.**

Fourier Analysis of Continuous-Time Signals

Periodic Signals

- We define a continuous-time signal $x(t)$ to be periodic if there is a positive nonzero value of T for which

$$x(t+T) = x(t) \qquad all\ t$$

- Two basic examples of periodic signals are the real sinusoidal signal

$$x(t) = \cos(\omega_0 t + \phi)$$

- and **the complex exponential signal**

$$x(t) = e^{j\omega_0 t}$$

- where $\omega_0 = 2\pi / T_0 = 2\pi f_0$ is called the **fundamental angular frequency**

Fourier Analysis of Continuous-Time Signals

Complex Exponential Fourier Series Representation

- **The complex exponential Fourier series representation of a periodic signal $x(t)$ with fundamental period T_0 is given by**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

- where c_k are known as the **complex Fourier coefficients** and are given by

$$c_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

Fourier Analysis of Continuous-Time Signals

Trigonometric Fourier Series

- **The trigonometric Fourier series representation of a periodic signal *x*(*t*) with fundamental period T_0, is given by**:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos k\omega_0 t + b_k \sin k\omega_0 t \right) \qquad \omega_0 = \frac{2\pi}{T_0}$$

- a_k, and b_k, are the Fourier coefficients given by:

$$a_k = \frac{2}{T_0} \int_{T_0} x(t) \cos k\omega_0 t dt$$

$$b_k = \frac{2}{T_0} \int_{T_0} x(t) \sin k\omega_0 t dt$$

Fourier Analysis of Continuous-Time Signals

Power Content of a Periodic Signal

- **The average power of a periodic signal *x*(*t*) over any period is given by**:

$$P = \frac{1}{T_0} \int_{T_0} \left| x(t) \right|^2 dt$$

- If *x*(*t*) is represented by the complex exponential Fourier series, then it can be shown that

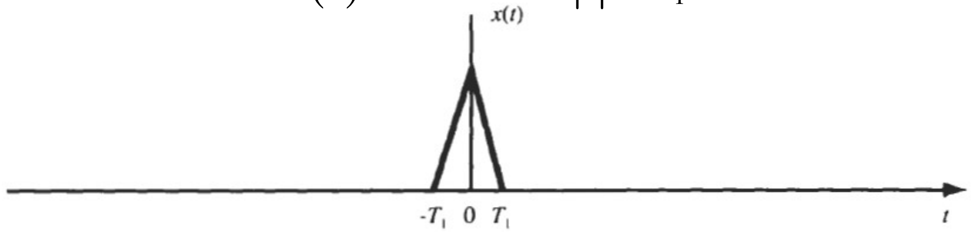
$$\frac{1}{T_0} \int_{T_0} \left| x(t) \right|^2 dt = \sum_{k=-\infty}^{\infty} \left| c_k \right|^2$$

Fourier Analysis of Continuous-Time Signals

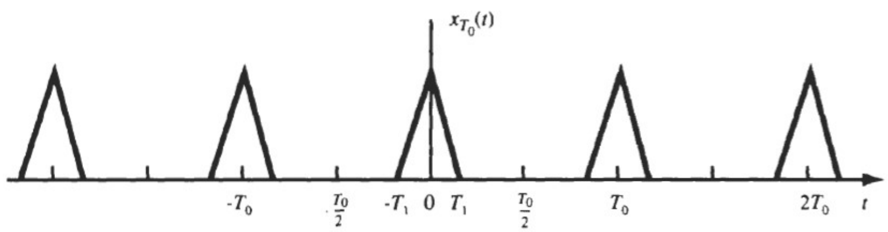
From Fourier Series to Fourier Transform

- Let $x(t)$ be a nonperiodic signal of finite duration:

$$x(t) = 0 \qquad |t| > T_1$$



- Let $x_{T_0}(t)$ be a periodic signal formed by repeating $x(t)$ with fundamental period T_0,



- If we let $T_0 \to \infty$, we have $\lim_{T_0 \to \infty} x_{T_0}(t) = x(t)$

Fourier Analysis of Continuous-Time Signals

- The complex exponential Fourier series of $x_{T_0}(t)$ is:

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0} \tag{10}$$

- where

$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \tag{10a}$$

- Since $x_{T_0}(t) = x(t)$ for $|t| < T_0/2$ and also since $x(t) = 0$ outside this interval, Eq. (10a) can be rewritten as:

$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt$$

Fourier Analysis of Continuous-Time Signals

- Let us define $X(\omega)$ as

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad (11)$$

- The complex Fourier coefficients c_k, can be expressed as

$$c_k = \frac{1}{T_0} X(k\omega_0) \qquad (11a)$$

- **<u>Substituting Eq. (11a) into Eq. (10), we have</u>**

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} X(k\omega_0) e^{jk\omega_0 t}$$

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\omega_0) e^{jk\omega_0 t} \omega_0 \qquad (12)$$

Fourier Analysis of Continuous-Time Signals

- As $T_0 \to \infty, \omega_0 = 2\pi/T_0$ becomes infinitesimal $(\omega_0 \to 0)$.
- Thus, $\omega_0 = \Delta\omega$ then Eq. (12) becomes

$$x_{T_0}(t)\Big|_{T_0 \to \infty} \to \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\Delta\omega) e^{jk\Delta\omega t} \Delta\omega \qquad (13)$$

- Therefore,

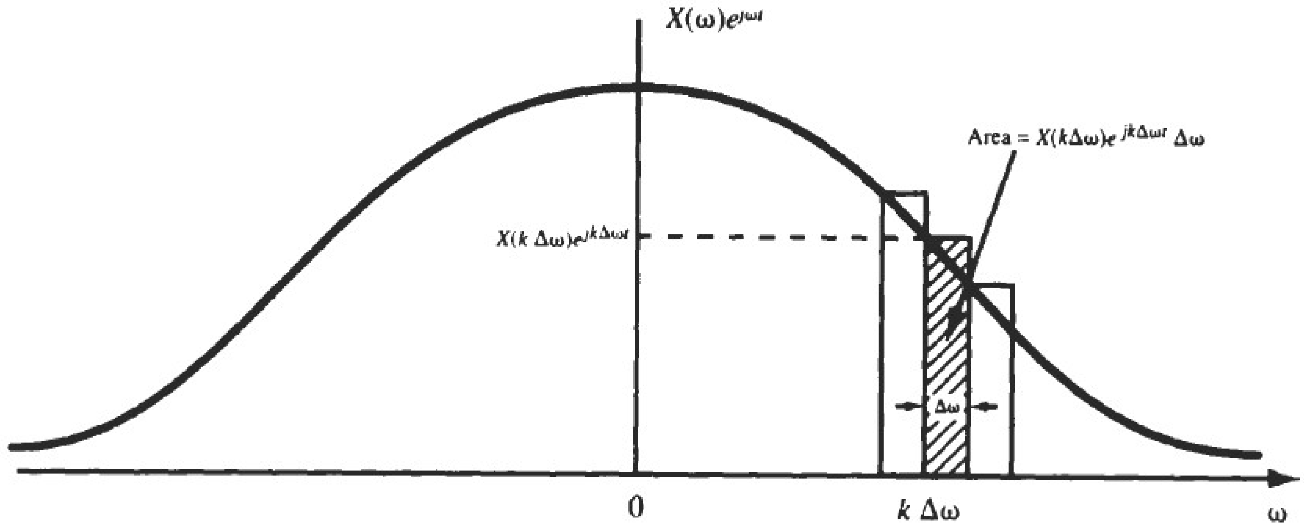
$$x(t) = \lim_{T_0 \to \infty} x_{T_0}(t) = \lim_{\Delta\omega \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\Delta\omega) e^{jk\Delta\omega t} \Delta\omega \qquad (14)$$

Fourier Analysis of Continuous-Time Signals

- **The sum on the right-hand side of Eq. (14) can be viewed as the area under the function $X(\omega)e^{j\omega t}$ as shown in the following figure**.
- Therefore, we obtain

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \qquad (15)$$

- **It's the Fourier representation of a nonperiodic *x*(*t*).**



Fourier Analysis of Continuous-Time Signals

Fourier Transform Pair

- The function $X(\omega)$ is called the Fourier transform of $x(t)$, and Eq. (15) defines the inverse Fourier transform of $X(\omega)$.

$$X(\omega) = \mathscr{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \mathscr{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

- **If impulse functions are permitted in the transform, some periodic signals (i.e., sin(*x*), cos(*x*)) can have Fourier transforms.**

Fourier Analysis of Continuous-Time Signals

Fourier Spectra

- **The Fourier transform $X(\omega)$ of $x(t)$ is, in general, complex, and it can be expressed as**:

$$X(\omega) = |X(\omega)| e^{j\phi(\omega)}$$

- The quantity $|X(\omega)|$ is called the magnitude spectrum of *x*(*t*), and $\phi(\omega)$ is called the phase spectrum of *x*(*t*).
- If *x*(*t*) is a real signal, we get $X(-\omega) = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt$
- Then it follows that

$$X(-\omega) = X^*(\omega)$$

$$|X(-\omega)| = |X(\omega)| \qquad \phi(-\omega) = -\phi(\omega)$$

- The amplitude spectrum $|X(\omega)|$ is an even function and the phase spectrum $\phi(\omega)$ is an odd one of ω.

Discrete Fourier Series

- A discrete-time signal (or sequence) $x[n]$ is periodic if there is a positive integer N for which

$$x[n+N] = x[n] \qquad all \;\; n$$

- The fundamental period N_0 of $x[n]$ is the smallest positive integer N.
- The complex exponential sequence

$$x[n] = e^{j(2\pi/N_0)n} = e^{j\Omega_0 n}$$

- where $\Omega_0 = 2\pi / N_0$, is a periodic sequence with fundamental period N_0.

Discrete Fourier Series

- One very important distinction between the discrete-time and the continuous-time complex exponential is that **the signals $e^{j\omega_0 t}$ are distinct for distinct values of ω_0 but the sequences $e^{j\Omega_0 n}$ which differ in frequency by a multiple of 2π, are identical.**
- Let $\Psi_k[n] = e^{jk\Omega_0 n}, \quad \Omega_0 = \dfrac{2\pi}{N_0}, \quad k = 0, \pm 1, \pm 2, \ldots$
- We have $\Psi_0[n] = \Psi_{N_0}[n] \quad \Psi_1[n] = \Psi_{N_0+1}[n] \quad \ldots \quad \Psi_k[n] = \Psi_{N_0+k}[n] \quad \ldots$
- In case of $\hat{\Omega}_0 = \Omega_0 + 2\pi k,\ e^{j\hat{\Omega}_0 n} = e^{j\Omega_0 n} e^{j2\pi kn} = e^{j\Omega_0 n}$ because $e^{j2\pi kn}$ is always 1: note that *k* and *n* are integer.
- However, incase of $e^{jk\omega_0 t} = e^{jk\frac{2\pi}{T_0}t} = e^{j\frac{2\pi}{T_0/k}t}, \quad k = 0, \pm 1, \pm 2, \ldots, \quad e^{jk\omega_0 t}$ are differ: note that *t* is not integer.

Discrete Fourier Series

- **The discrete Fourier series representation of a periodic sequence $x[n]$ with fundamental period N_0 is given by**

$$x[n] = \sum_{k=0}^{N_0-1} c_k e^{jk\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N_0}$$

Note that the range of k is [0 N_0-1] (see page 33)

- where c_k are the Fourier coefficients and given by:

$$c_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-jk\Omega_0 n}$$

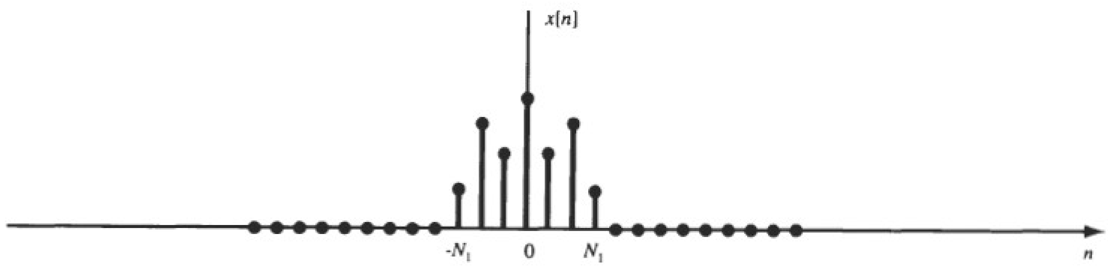
- **The DFS coefficients are periodic**.

From Discrete Fourier Series to Fourier Transform (DTFT)

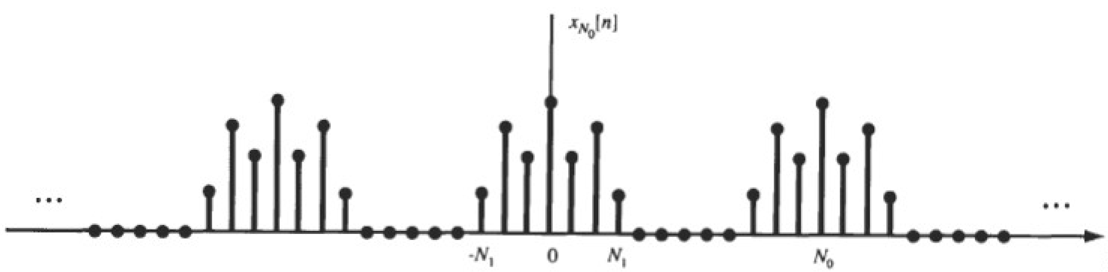
- Let $x[n]$ be a nonperiodic sequence of finite duration: that is, for some positive integer N_1,

$$x[n] = 0 \qquad |n| > N_1$$

- Such a sequence is shown in the following figure.



- Let $x_{N_0}[n]$ be a periodic sequence formed by repeating $x[n]$ with fundamental period N_0 as shown in the following figure.



From Discrete Fourier Series to Fourier Transform (DTFT)

- If we let $N_0 \to \infty$, we have $\lim_{N_0 \to \infty} x_{N_0}[n] = x[n]$.
- **The discrete Fourier series of $x_{N_0}[n]$ is given by**

$$x_{N_0}[n] = \sum_{k=\langle N_0 \rangle} c_k e^{jk\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N_0} \qquad (16)$$

$$c_k = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x_{N_0}[n] e^{-jk\Omega_0 n} \qquad (17)$$

- Since $x_{N_0}[n] = x[n]$ for $|n| \le N_1$ and also since $x[n]=0$ outside this interval, Eq. (17) can be rewritten as

$$c_k = \frac{1}{N_0} \sum_{n=-N_1}^{N_1} x[n] e^{-jk\Omega_0 n} = \frac{1}{N_0} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\Omega_0 n}$$

- Let us define $X(\Omega)$ as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

where n is [0,1,2,3,.. N_0-1,…] and fundamental angular frequency in Ω is Ω_0, $exp[-j\Omega n]$ → $exp[-j\Omega_0 n]$. So $X(\Omega)$ has 2π period (see page 33).

From Discrete Fourier Series to Fourier Transform (DTFT)

- The Fourier coefficients c_k can be expressed as:

$$c_k = \frac{1}{N_0} X\left(k\Omega_0\right) \qquad (18)$$

- Substituting Eq. (18) into Eq. (16), we have

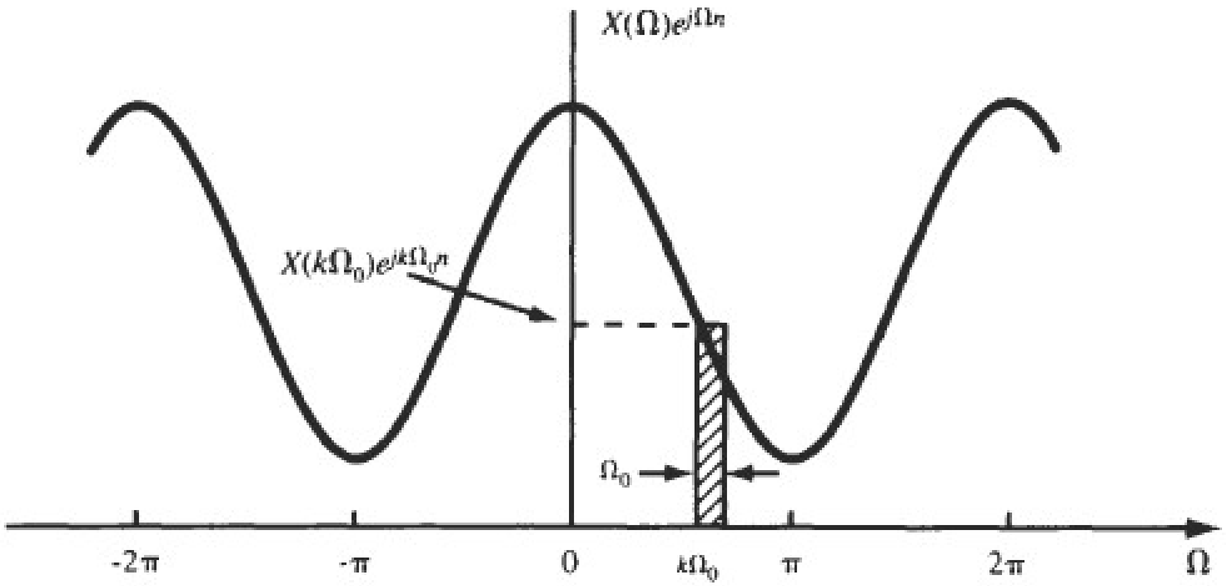
$$x_{N_0}\left[n\right] = \frac{1}{2\pi} \sum_{k=\langle N_0 \rangle} X\left(k\Omega_0\right) e^{jk\Omega_0 n} \Omega_0 \qquad (19)$$

- **$X(\Omega)$ is periodic with period 2*π* and so is $e^{j\Omega n}$.**
- **So, the product $X(\Omega)e^{j\Omega n}$ is periodic with period 2*π*.**

where *n* is [0,1,2,3,.. N_0-1,…] and fundamental angular frequency in Ω is Ω_0, *exp*[-$j\Omega n$] → *exp*[-$j\Omega_0 n$]. So $X(\Omega)$ has 2*π* period (see page 33).

From Discrete Fourier Series to Fourier Transform (DTFT)

- As shown in the following figure, each term in the summation in Eq. (19) represents the area of a rectangle of height $X(k\Omega_0)e^{jk\Omega_0 n}$ and width Ω_0.



From Discrete Fourier Series to Fourier Transform (DTFT)

- As $N_0 \to \infty$, $\Omega_0 = 2\pi / N_0$, becomes infinitesimal ($\Omega_0 \to 0$) and Eq. (19) passes to an integral.
- Furthermore, since the summation in Eq. (19) is over N_0, consecutive intervals of width $\Omega_0 = 2\pi / N_0$, the total interval of integration will always have a width 2π.
- Thus, as $N_0 \to \infty$ Eq. (19) becomes

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega \qquad (20)$$

From Discrete Fourier Series to Fourier Transform (DTFT)

Fourier Transform Pair

- The function $X(\omega)$ is called the discrete-time Fourier transform (DTFT) of $x[n]$, and equation (20) defines the inverse Fourier transform of $X(\omega)$.

$$X(\Omega) = \mathscr{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

$$x[n] = \mathscr{F}^{-1}\{X(\Omega)\} = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega$$

From Discrete Fourier Series to Fourier Transform (DTFT)

Fourier Spectra

- The Fourier transform $X(\omega)$ of $x[n]$ is, in general, complex and can be expressed as:

$$X(\Omega) = |X(\Omega)| e^{j\phi(\Omega)}$$

- The quantity $|X(\Omega)|$ is called the magnitude spectrum of $x[n]$, and $\phi(\Omega)$ is called the phase spectrum of $x[n]$.

- Furthermore, if $x[n]$ is real, the amplitude spectrum $|X(\Omega)|$ is an even function and the phase spectrum $\phi(\Omega)$ is an odd function.

Frequency Response of Discrete-Time LTI Systems

- The output $y[n]$ of a discrete-time LTI system equals the convolution of the input $x[n]$ with the impulse response $h[n]$; that is,

$$y[n] = x[n] * h[n]$$

- Applying the convolution property, we obtain

$$Y(\Omega) = X(\Omega)H(\Omega)$$

- where $Y(\Omega)$, $X(\Omega)$, and $H(\Omega)$ are the Fourier transforms of $y[n]$, $x[n]$, and $h[n]$, respectively.
- **Unlike the frequency response of continuous-time systems, that of all discrete-time LTI systems is periodic with period 2π.**

$$H(\Omega) = H(\Omega + 2\pi)$$

Sampling

- **Most discrete-time signals come from sampling a continuous-time signal, such as speech, audio signals, radar, sonar data, and biological signals.**
- **Sampling theory gives precise conditions under which an analog signal may be uniquely represented in terms of its samples.**
- When implementing holography simulations on the computer we need to represent functions by sampled values and apply transform/processing methods designed for these discrete signals.
- It would be great to model the physical elements with many samples.

Sampling

- Consider the two-dimensional (2D) analytic function $g(x,y)$ and suppose it is sampled in a uniform manner (see Fig. 2.1) in the x and y directions, which is indicated by

$$g(x,y) \rightarrow g(m\Delta x, n\Delta y).$$

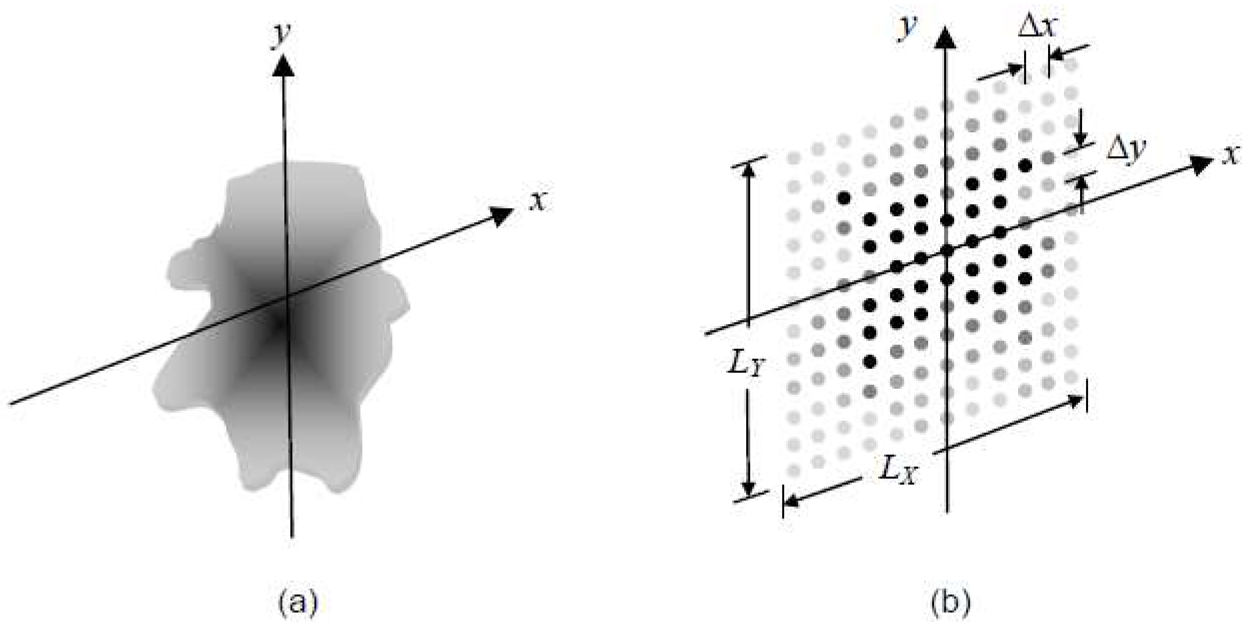


Figure 2.1 Two-dimensional function: (a) analytic and (b) sampled versions.

Sampling

- Sample interval is Δx in the *x* direction and Δy in the *y* direction, and *m* and *n* are integer-valued indices of the samples: **the respective sample rates (sampling frequency) are $1/\Delta x$ and $1/\Delta y$**.
- In practice, the sampled space is finite and, assuming it is composed of $M \times N$ samples in the *x* and *y* directions, respectively, *m* and *n* are defined with the following values:

$$m = -\frac{M}{2}, \ldots, \frac{M}{2} - 1, \qquad n = -\frac{N}{2}, \ldots, \frac{N}{2} - 1.$$

- This is a standard index arrangement where *M* and *N* are assumed to be even.

Sampling

- A finite physical area spanned by the sampled space, and this is given by $L_X \times L_Y$, where L_X is the length along the *x* side of the sampled space and L_Y is the length along the *y* side (see Fig. 2.1).
- They represent physical distances and are related to the sampling parameters by

$$L_X = M\Delta x, \qquad L_Y = N\Delta y.$$

- **If D_X is the support in the *x* direction and D_Y is the support in the *y* direction, then for the significant values of $g(x,y)$ to be contained within the array requires**

$$D_X < L_X, \qquad D_Y < L_Y.$$

Sampling

- **Another concern is whether the sample intervals are small enough to preserve features of *g*(*x*,*y*).**

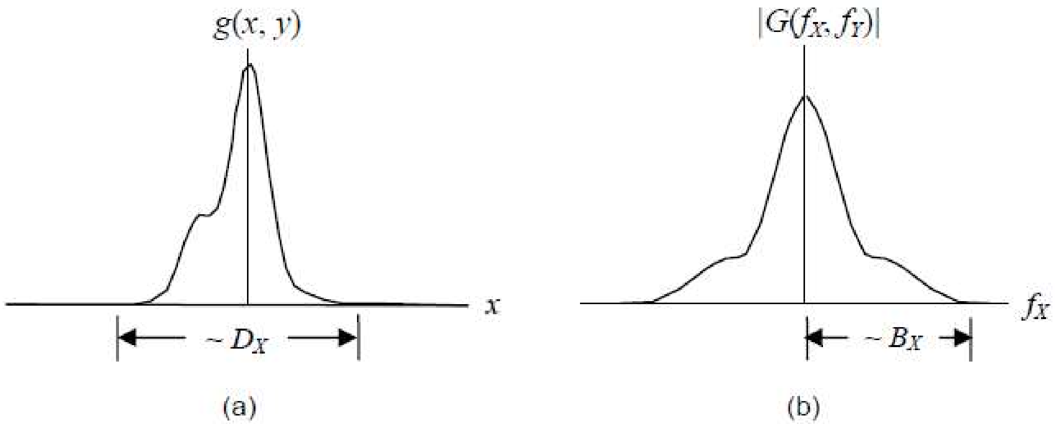


Figure 2.2 Illustration of the (a) support D_X and (b) bandwidth B_X along the x axis of $g(x, y)$. Bandwidth is commonly defined as a half-width measure and is illustrated here with a profile of $|G(f_X, f_Y)|$, the Fourier transform magnitude of $g(x, y)$.

- If functions that are bandlimited, where the spectral content of the signal is limited to a finite range of frequencies, **a continuous function can be recovered exactly from the samples if the sample interval is smaller than a specific value**.

Sampling

- **The Shannon–Nyquist sampling theorem, extended to two dimensions, states this requirement as**:

$$\Delta x < \frac{1}{2B_X}, \qquad \Delta y < \frac{1}{2B_Y} \tag{21}$$

- where B_X is the bandwidth of the spectrum of the continuous function along the *x* direction and B_Y is the bandwidth along the *y* direction.
- Bandwidth is illustrated in Fig. 2.2(b).
- **Violating Eq. (21) results in aliasing, in which undersampled high-frequency components in the signal are interpreted erroneously as low frequency content**.

Periodic Sampling

- **Discrete-time signals are formed by periodically sampling a continuous-time signal**

$$x(n) = x_a(nT_s)$$

- **The sample spacing T_s is the sampling period, and $f_s = 1/T_s$ is the sampling frequency in samples per second.**
- The continuous-time signal is first multiplied by a periodic sequence of impulses,

$$s_a(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

- To form the sampled signal

$$x_s(t) = x_a(t)s_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s)$$

Periodic Sampling

- The sampled signal is converted into a discrete-time signal by mapping the impulses that are spaced in time by T_s into a sequence $x(n)$ where the sample values are indexed by the integer variable n:

$$x(n) = x_a(nT_s)$$

- This process is illustrated in Fig. (b).

(b)

- **<u>The Fourier transform of $\delta(t - nT_s)$ is $e^{-jn\Omega T_s}$, so the Fourier transform of the sampled signal $x_s(t)$ is</u>**

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_a(nT_s) e^{-jn\Omega T_s} \tag{1}$$

Examples

- **The Fourier transform of $\delta(t)$ is:**

$$\mathscr{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = 1$$

- Because $\int_{-\infty}^{\infty} \phi(t)\delta(t)\, dt = \phi(0)$

- **The Fourier transform of $\delta(t - nT_s)$ is $e^{-jn\Omega T_s}$.**

Periodic Sampling

- Another expression for $X_s(j\Omega)$ follows by noting that the Fourier transform of $s_a(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$ is

$$S_a(j\Omega)=\frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega_s)$$

- where $\Omega_s=2\pi/T_s$ is the sampling frequency in radians per second.
- By multiplication property of Fourier transform,

$$X_s(j\Omega)=\sum_{n=-\infty}^{\infty}x_a(nT_s)e^{-jn\Omega T_s}=\frac{1}{2\pi}X_a(j\Omega)*S_a(j\Omega)=\frac{1}{T_s}\sum_{k=-\infty}^{\infty}X_a(j\Omega-jk\Omega_s)$$

- **Note that the above equation is derived from the Fourier analysis of the continuous time signal (see pages 55 to 59).**

Periodic Sampling

- Finally, the discrete-time Fourier transform of $x(n)$:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\omega} = \sum_{n=-\infty}^{\infty} x_a(nT_s) e^{-jn\omega} \quad (2)$$

$X\left(e^{j\omega}\right)$ **is periodic with period 2π (see page 37)**

- Comparing Eq. (1) with Eq. (2), it follows that:
  (if ω in Eq.(2), set to ΩT_s, Eqs (1) and (2) will be same)

$$X\left(e^{j\omega}\right) = X_s(j\Omega)\Big|_{\Omega=\omega/T_s} = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a\left(j\frac{\omega}{T_s} - j\frac{2\pi k}{T_s}\right) \quad (3)$$

- **Thus, $X(e^{j\omega})$ is a frequency-scaled version of $X_s(j\Omega)$, with the scaling defined by**

$$\omega = \Omega T_s$$

- This scaling, which makes $X(e^{j\omega})$ periodic with a period of 2π, is a consequence of the time-scaling that occurs when $x_s(t)$ is converted to $x(n)$.
  (In Eq. (3), $X_a\left(j\frac{\omega}{T_s} - j\frac{2\pi k}{T_s}\right)$ has 2π period)

Note that the range of ω is [0 2π], $2\pi/T_s$ is sampling frequency

Sampling Theorem

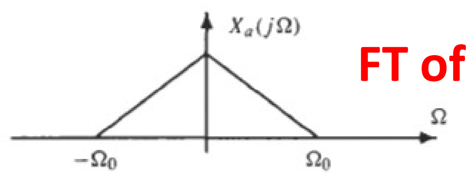
- If $x_a(t)$ is strictly bandlimited,

$$X_a(j\Omega) = 0 \qquad |\Omega| > \Omega_0$$

- Then $x_a(t)$ may be uniquely recovered from its sampled signal $x_s(t)$ if

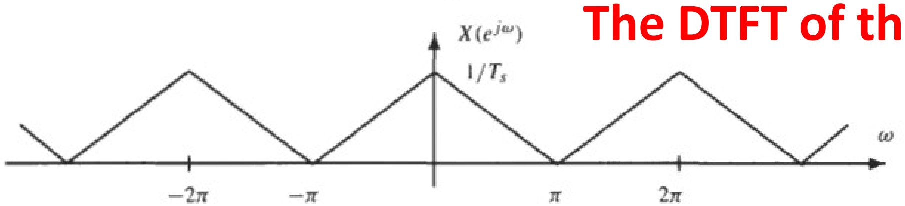
$$\Omega_s = \frac{2\pi}{T_s} \geq 2\Omega_0$$

- The frequency Ω_0 is called the Nyquist frequency, and the minimum sampling frequency, $\Omega_s = 2\Omega_0$, is called the Nyquist rate.



(a)

FT of the continuous time signal



(b)

The DTFT of the sampled sequence *x*[*n*]

Examples

Find the Fourier transform of the periodic impulse train of $\delta(t-nT_s)$:

$$\delta_{T_0}(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT_0)$$

- From the inverse Fourier transform definition, we have

$$\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega=2\pi x(t)$$

- Changing *t* to -*t*, we obtain

$$\int_{-\infty}^{\infty}X(\omega)e^{-j\omega t}d\omega=2\pi x(-t)$$

- Now interchanging *t* and ω, we get

$$\int_{-\infty}^{\infty}X(t)e^{-j\omega t}dt=2\pi x(-\omega)$$

- Since

$$\mathscr{F}\{X(t)\}=\int_{-\infty}^{\infty}X(t)e^{-j\omega t}dt$$

Examples

- We get the following duality property:

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

- We know that the Fourier transform of $\delta(t)$ is 1.
- Thus, by the duality property we get

$$1 \leftrightarrow 2\pi\delta(-\omega) = 2\pi\delta(\omega)$$

- And, we know that $x(t) = \mathscr{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$
- We can derive the following equation:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}\int_{-\infty}^{\infty}\delta(0)d\omega = e^{j\omega_0 t}$$

- Then we get

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0) \quad (4)$$

Examples

- We express periodic signal $x(t)$ as

$$x(t)=\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0=\frac{2\pi}{T_0}$$

- Taking the Fourier transform of both sides and using Eq. (2) and the linearity property, we get

$$X(\omega)=2\pi\sum_{k=-\infty}^{\infty} c_k \delta(\omega-k\omega_0)$$

- The complex exponential Fourier series of $\delta_{T_0}(t)$ is given by:

$$\delta_{T_0}(t)=\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0=\frac{2\pi}{T_0}$$

- The Fourier coefficients can be obtained since $\delta(t)$ is involved:

$$c_k=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\delta(t)e^{-jk\omega_0 t}dt=\frac{1}{T_0}$$

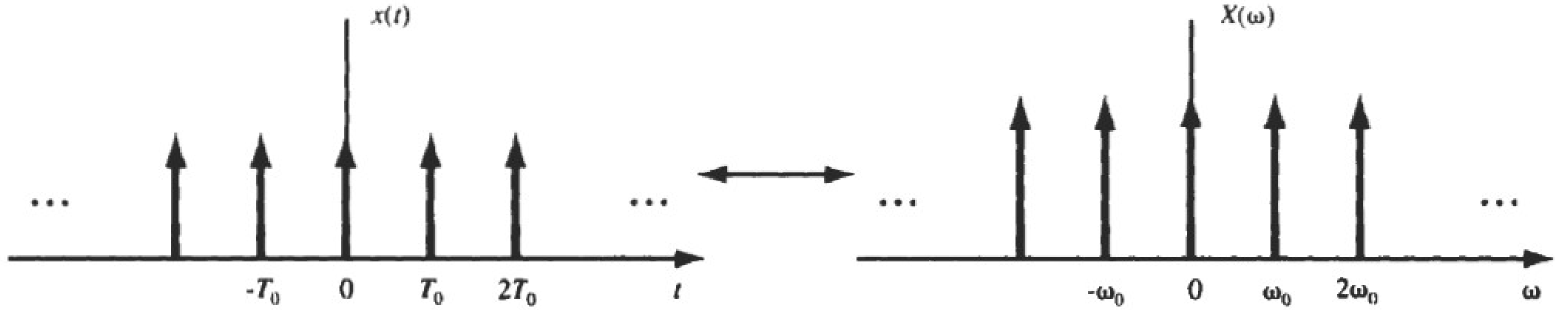
Examples

- Finally, we get

$$\mathscr{F}\left[\delta_{T_0}(t)\right] = \frac{2\pi}{T_0}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_0)$$

$$= \omega_0 \sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_0)$$

- Thus, the Fourier transform of a unit impulse train is also a similar impulse train.



Examples: Convolution Calculation

- By convolution definition, we get

$$
\begin{aligned}
x(t) * \delta(t-t_0) &= \delta(t-t_0) * x(t) \\
&= \int_{-\infty}^{\infty} \delta(\tau - t_0) x(t-\tau) d\tau = x(t-\tau)\big|_{\tau = t_0} \\
&= x(t-t_0)
\end{aligned}
$$

Discrete Fourier Transform

- We have seen how to represent a sequence in terms of a linear combination of complex exponentials using the discrete-time Fourier transform (DTFT).
- **For finite-length sequences there is another representation, called the discrete Fourier transform (DFT)**.
- Unlike the DTFT, which is a continuous function of a continuous variable, ω**,** **the DFT is a sequence that corresponds to samples of the DTFT.**
- Such a representation is very useful for digital computations.

Discrete Fourier Transform

- **The DFT may be easily developed from the discrete Fourier series for periodic sequences.**
- Let $x(n)$ be a finite-length sequence of length N that is equal to zero outside the interval [0, N-1].
- A periodic sequence $\tilde{x}(n)$ may be formed from $x(n)$ as follows:

$$\tilde{x}(n) = \sum_{k=-\infty}^{\infty} x\left(n + kN\right)$$

- **A periodic sequence may be expanded using the DFS as Eq. (1).**

$$\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}\left(k\right) e^{j2\pi nk/N} \qquad (1)$$

Discrete Fourier Transform

- Fourier series coefficients, $\tilde{X}(k)$ may be derived by multiplying both sides of this expansion by $e^{-j2\pi nl/N}$ **summing over one period and using the fact that the complex exponentials are orthogonal**:

$$\sum_{k=0}^{N-1} e^{j2\pi n(k-l)/N} = \begin{cases} N & k=l \\ 0 & k \neq l \end{cases}$$

- The result is

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j2\pi nk/N}$$

Discrete Fourier Transform

- Because $x(n)=\tilde{x}(n)$ for n = 0, 1, . . . ,N-1, $x(n)$ may similarly be expanded as follows:

$$x(n)=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}(k)e^{j2\pi nk/N} \qquad 0\le n<N$$

$$\tilde{X}(k)=\sum_{n=0}^{N-1}x(n)e^{-j2\pi nk/N} \qquad 0\le k<N$$

- **Because the DFS coefficients are periodic, if we let *X*(*k*) be one period of $\tilde{X}(k)$ and replace $\tilde{X}(k)$ in the sum with *X*(*k*), then we have**

$$x(n)=\frac{1}{N}\sum_{k=0}^{N-1}X(k)e^{j2\pi nk/N} \qquad 0\le n<N$$

- The sequence $X(k)$ is called the N-point DFT of $x(n)$.

Discrete Fourier Transform

- These coefficients are related to $x(n)$ as follows:

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \qquad 0 \le k < N$$

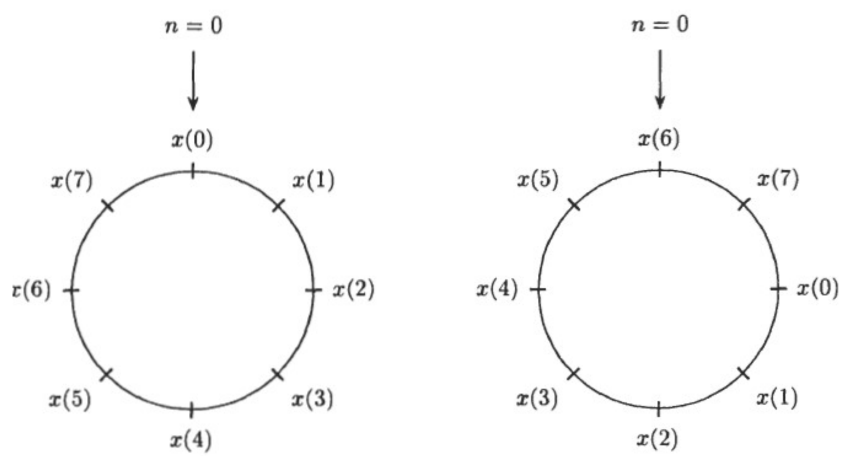
- **Comparing the definition of the DFT of *x*(*n*) to the DTFT, it follows that the DFT coefficients are samples of the DTFT:**

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi nk/N} = X\left(e^{j\omega}\right)\Big|_{\omega=2\pi k/N}$$

- **Note that DFT has a periodic property so that we extract the values in the range of [0, *N*-1].**

DFT Circular Shift Property

- **<u>The DFT has a circular shift property.</u>**
- Suppose that the values of a sequence *x*(*n*), from *n*=0 to *n*=*N*-1, are marked around a circle as illustrated in Figure (a).



(a) An eight-point sequence. (b) Circular shift by two.

- A circular shift to the right by n_o corresponds to a rotation of the circle n_o positions in a clockwise direction as shown in Figure (b).

DFT Circular Shift Property

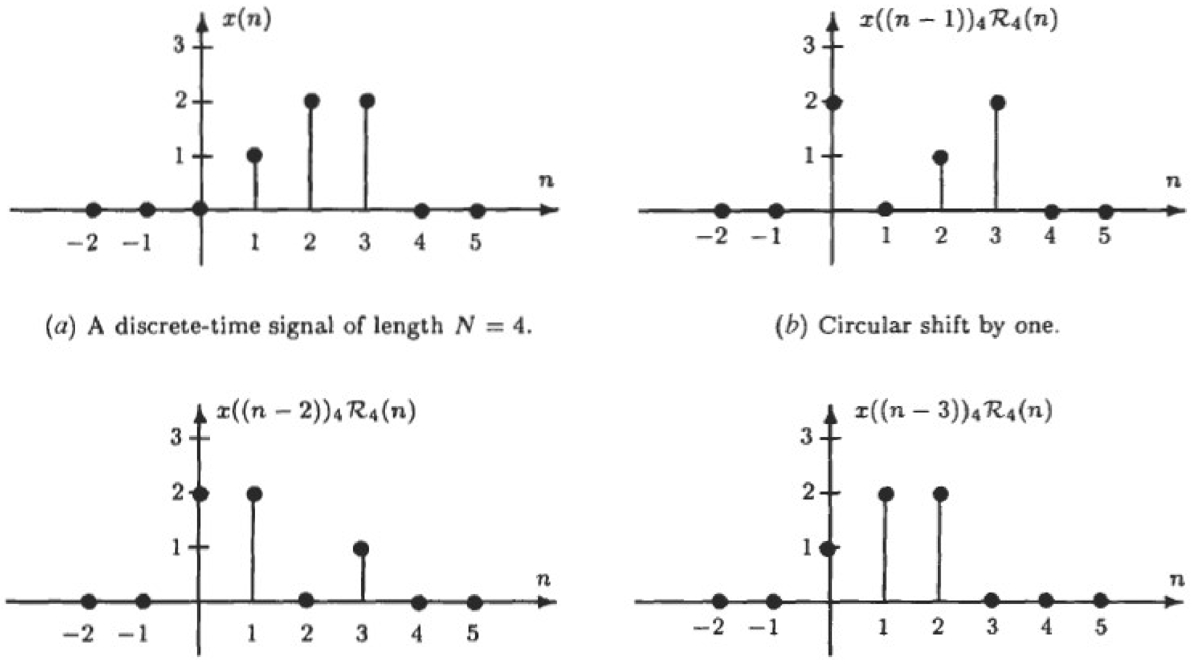
- Another way to circularly shift a sequence is to form the periodic sequence $\tilde{x}(n)$, perform a linear shift, $\tilde{x}(n-n_0)$ and then extract one period of $\tilde{x}(n-n_0)$ by multiplying by a rectangular window.
- **The circular shift of a sequence *x*(*n*) is defined as follows:**

$$x((n-n_0))_n \mathcal{R}_N(n) = \tilde{x}(n-n_0)\mathcal{R}_N(n)$$

- where n_o is the amount of the shift and $\mathcal{R}_N(n)$ is a rectangular window.

DFT Circular Shift Property

- **<u>Examples illustrating the circular shift of a four-point sequence are shown in Figures (a), (b). (c), and (d).</u>**



(a) A discrete-time signal of length $N = 4$.

(b) Circular shift by one.

(c) Circular shift by two.

(d) Circular shift by three.

DFT Circular Convolution

- Let $h(n)$ and $x(n)$ be finite-length sequences of length N with N-point DFTs $H(k)$ and $X(k)$, respectively.
- The sequence that has a DFT equal to the product $Y(k)=H(k)X(k)$ is

$$y(n)=\left[\sum_{k=0}^{N-1}\tilde{h}(k)\tilde{x}(n-k)\right]\mathcal{R}_N(n)=\left[\sum_{k=0}^{N-1}\tilde{h}(n-k)\tilde{x}(k)\right]\mathcal{R}_N(n) \qquad (2)$$

- where $\tilde{x}(n)$ and $\tilde{h}(n)$ are the periodic extensions of the sequences $x(n)$ and $h(n)$, respectively.

DFT Circular Convolution

- Because $\tilde{h}(n) = h(n)$ for $0 \leq n < N$, the sum in Eq. (2) may also be written as

$$y(n) = \left[\sum_{k=0}^{N-1} h(k) \tilde{x}(n-k) \right] \mathcal{R}_N(n) \quad (3)$$

- <u>**The sequence *y*(*n*) in Eq. (3) is the *N*-point circular convolution of *h*(*n*) with *x*(*n*), and it is written as**</u>

$$y(n) = h(n) \; Ⓝ \; x(n) = x(n) \; Ⓝ \; h(n)$$

Circular Versus Linear Convolution

- **In general, circular convolution is not the same as linear convolution.**
- However, there is a relationship between circular and linear convolution that illustrates what steps must be added to ensure that they are the same.
- Specifically, let *x*(*n*) and *h*(*n*) be finite-length sequences and let *y*(*n*) be the linear convolution:

$$y(n) = x(n) * h(n)$$

- The *N*-point circular convolution of *x*(*n*) with *h*(*n*) is related to *y*(*n*) as follows:

$$h(n) \, Ⓝ \, x(n) = \left[\sum_{k=-\infty}^{\infty} y(n+kN) \right] \mathcal{R}_N(n) \qquad (4)$$

- **The circular convolution of two sequences is found by performing the linear convolution and aliasing the result.**

Circular Versus Linear Convolution

Example: Let us find the four-point circular convolution of the sequences *h*(*n*) and *x*(*n*): the linear convolution is $y(n)=\delta(n)+\delta(n-1)+2\delta(n-2)+2\delta(n-3)+3\delta(n-5)$

- We may set up a table to evaluate the sum

$$h(n) \; \textcircled{N} \; x(n)=\left[\sum_{k=-\infty}^{\infty} y(n+kN)\right]\mathcal{R}_N(n)$$

- This is done by listing the values of the sequence *y*(*n* + *kN*) in a table and summing these values for *n* = 0,1,2,3.

- Thus, we have

n	0	1	2	3	4	5	6	7
$y(n)$	1	1	2	2	0	3	0	0
$y(n+4)$	0	3	0	0	0	0	0	0
$h(n) \textcircled{4} x(n)$	1	4	2	2	–	–	–	–

- Summing the columns for $0\le n\le 3$, we have

$$h(n)\textcircled{4}x(n)=\delta(n)+4\delta(n-1)+2\delta(n-2)+2\delta(n-3)$$

Circular Versus Linear Convolution

- An important property that follows from Eq. (4) is that if *y*(*n*) is of length *N* or less, circular convolution is equivalent to linear convolution.

$$h(n) \, Ⓝ \, x(n) = h(n) * x(n)$$

- **Thus, if *h*(*n*) and *x*(*n*) are finite-length sequences of length N_1 and N_2, respectively, $y(n) = h(n) * x(n)$ is of length $N_1 + N_2 - 1$, and the *N*-point circular convolution is equivalent to linear convolution provided $N \geq N_1 + N_2 - 1$.**

Linear Convolution Using DFT

- The DFT provides a convenient way to perform convolutions without the convolution sum.
- Specifically, if $h(n)$ is N_1 points long and $x(n)$ is N_2 points long, h(n) may be linearly convolved with $x(n)$ as follows:

  1. **Pad the sequences $h(n)$ and $x(n)$ with zeros so that they are of length $N \geq N_1 + N_2 - 1$.**
  2. Find the N-point DFTs of $h(n)$ and $x(n)$.
  3. Multiply the DFTs to form the product $Y(k)=H(k)X(k)$.
  4. Find the inverse DFT of $Y(k)$.

- **Significant computational savings for DFT may be realized with the fast Fourier transform (FFT).**

2D Fourier Transform Definitions

- The analytic Fourier transform of a function *g* of two variables *x* and *y* is given by:

$$G\left(f_X, f_Y\right) = \int_{-\infty}^{\infty}\int g\left(x, y\right)\exp\left[-j2\pi\left(f_X x + f_Y y\right)\right]dxdy,$$

- where $G(f_X, f_Y)$ is the transform result and f_X and f_Y are independent frequency variables associated with *x* and *y*, respectively.
- This operation is often described in a shorthand manner as $\Im\{g(x,y)\} = G(f_X, f_Y).$
- The analytic inverse Fourier transform is given by:

$$g\left(x, y\right) = \int_{-\infty}^{\infty}\int G\left(f_X, f_Y\right)\exp\left[j2\pi\left(f_X x + f_Y y\right)\right]df_X df_Y.$$

- The shorthand notation for this operation is

$$\Im^{-1}\{G(f_X, f_Y)\} = g(x,y).$$

Discrete Fourier Transform from Continuous Transform

- The analytic Fourier transform of a function *g* of two variables *x* and *y* is repeated for reference:

$$G(f_X, f_Y) = \int_{-\infty}^{\infty}\int g(x,y)\exp\left[-j2\pi\left(f_X x + f_Y y\right)\right] dxdy. \qquad (5)$$

- First, assume *g*(*x*,*y*) is sampled as $g(m\Delta x, n\Delta y) \rightarrow \tilde{g}(m,n)$.
- The integrals in Eq. (5) can be approximated using a Riemann sum:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ldots\, dxdy \rightarrow \sum_{n=-N/2}^{N/2-1}\sum_{m=-M/2}^{M/2-1} \ldots\, \Delta x \Delta y.$$

Discrete Fourier Transform from Continuous Transform

- **The convention for the frequency domain is to divide this continuous space indicated by f_X and f_Y, into *M* and *N* evenly spaced coordinate values as follows (*M*, *N*: total number of samples)**

$$f_X \rightarrow \frac{p}{M\Delta x}, \quad where \quad p = -\frac{M}{2}, \ldots, \frac{M}{2} - 1;$$
$$f_Y \rightarrow \frac{q}{N\Delta y}, \quad where \quad q = -\frac{N}{2}, \ldots, \frac{N}{2} - 1; \qquad (6)$$

- where *p* and *q* are integers that index multiples of the frequency sample intervals.

$$\Delta f_X = \frac{1}{M\Delta x} = \frac{1}{L_X}, \quad and \quad \Delta f_Y = \frac{1}{N\Delta y} = \frac{1}{L_Y}. \qquad (6\text{-}1)$$

- In fact, *p* and *q* take on the same values as *m* and *n*, respectively, since the spatial and frequency arrays have the same number of elements.

Discrete Fourier Transform from Continuous Transform

- **Note that the maximum absolute values of the frequency coordinates in Eq. (6) are the Nyquist frequencies $1/(2\Delta x) = f_{NX}$ and $1/(2\Delta y) = f_{NY}$.**
- Incorporating Eq. (6) into the complex exponential kernel of Eq. (5) yields

$$\exp\left[-j2\pi\left(f_X x + f_Y y\right)\right] \rightarrow \exp\left[-j2\pi\left(\frac{p}{M\Delta x} m\Delta x + \frac{q}{N\Delta y} n\Delta y\right)\right]$$

$$= \exp\left[-j2\pi\left(\frac{pm}{M} + \frac{qn}{N}\right)\right].$$

Discrete Fourier Transform from Continuous Transform

- Finally, we arrive at the following form of the DFT:

$$\tilde{G}(p,q) = \sum_{m=-M/2}^{M/2-1} \sum_{n=-N/2}^{N/2-1} \tilde{g}(m,n) \exp\left[-j2\pi\left(\frac{pm}{M} + \frac{qn}{N}\right)\right], \quad (7)$$

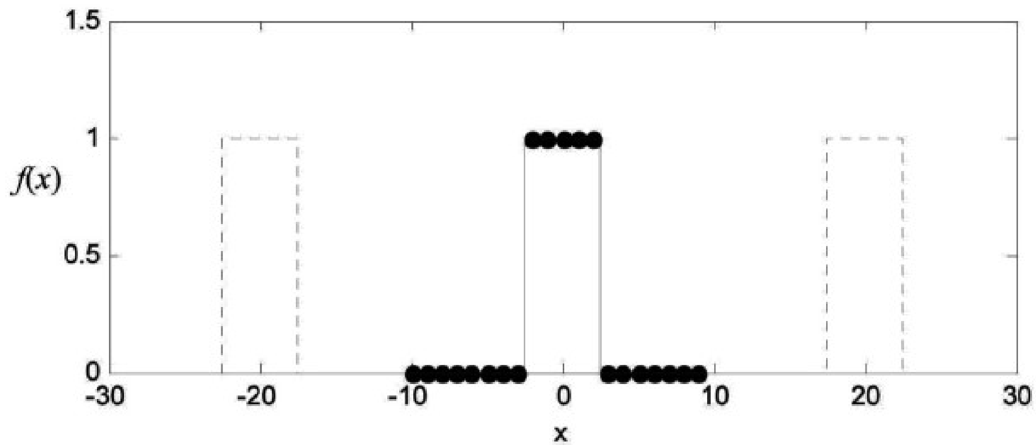
- The inverse discrete Fourier transform (DFT^{-1}) is derived in a similar way and is written as:

$$\tilde{g}(m,n) = \frac{1}{MN} \sum_{p=-N/2}^{M/2-1} \sum_{q=-M/2}^{N/2-1} \tilde{G}(p,q) \exp\left[j2\pi\left(\frac{pm}{M} + \frac{qn}{N}\right)\right]. \quad (8)$$

- **The forward and inverse DFTs are not usually accomplished with a direct use of Eqs. (7), (8): they are accomplished with the computationally efficient FFT and FFT^{-1} algorithms.**

Discrete Fourier Transform from Continuous Transform

- **Analytic rectangle function is shown in Fig. (a) (solid line) along with a sampled version (dots).**

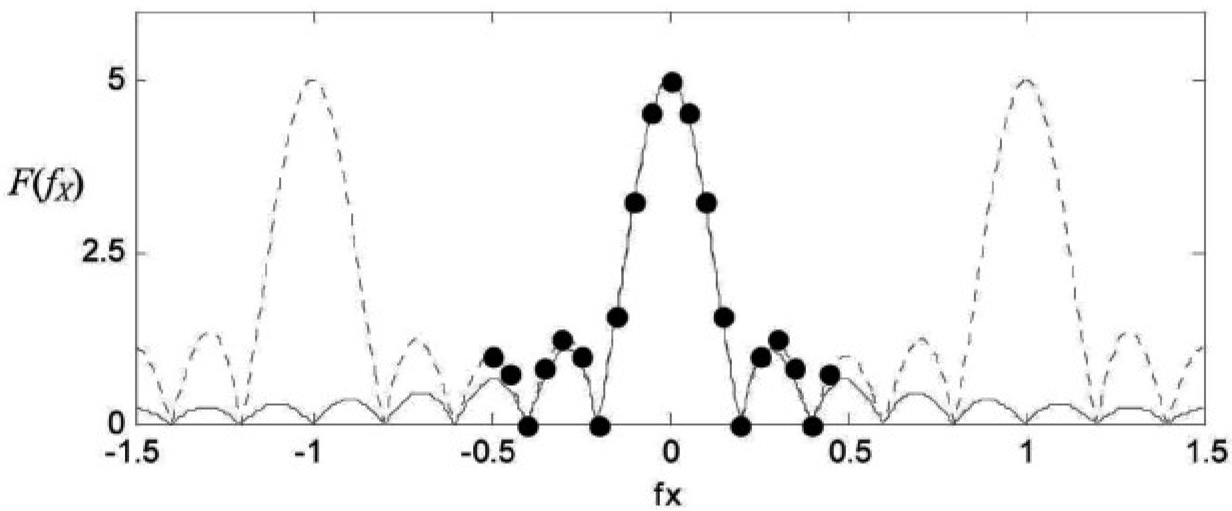


(a)

- **The periodic form of the function, which extends (virtually) beyond the original span of the sample vector, is also indicated (dashed line).**

Discrete Fourier Transform from Continuous Transform

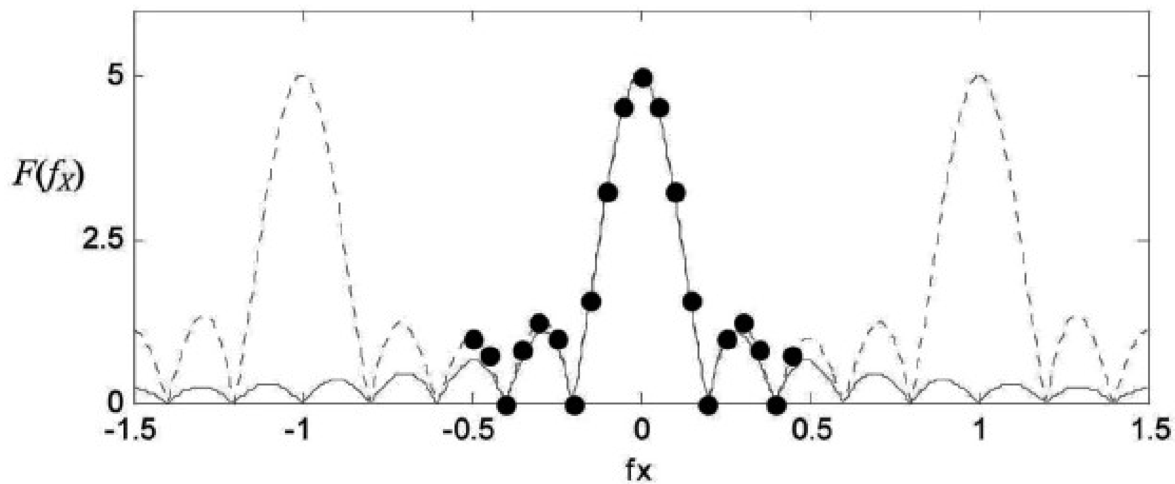
- Figure (b) shows the magnitude of the analytic spectrum of the rectangle (solid), the FFT result (dots), and the periodic spectrum (dashed).



(b)

Discrete Fourier Transform from Continuous Transform

- The most difference between the analytic and sample spectra in this case is slightly larger sample values in the magnitude at higher frequencies.
- **<u>This effect results from aliasing of under-sampled frequencies in the rectangle spectrum</u>.**



(b)

If the sampling frequency is $2\Omega_B$ where Ω_B is max frequency of the signal, the dashed line has the fx with 2π period